

Chapter Three. The Philosophy of Number

§ 13. The Fregean assignment of numbers to concepts.

We have advanced a view of concepts, which we believe is close to the dominant motivating view in the tradition of philosophy, according to which concepts are the senses of predicate expressions (or are obtained from the latter by abstracting from certain inessential differences of meaning from context to context). Thus concepts are taken to be that through which the individual cognizing subject articulates his experience of the world, and it follows that any epistemological shortfall between what is given in experience and the constitution of that world is to be traced solely to our lack of knowledge of the objects in the world (and of their properties, relations, processes, etc.). For a lack of knowledge concerning the concepts which we employ is, on this view, a meaningless one: if we express a belief that whales fall under the concept fish, we either display that we know too little about whales, or we reveal that we are employing a concept fish which is different from the biologically standard concept; what we cannot reveal is that we know too little about the concept fish which we are actually using. We have seen also that Frege defended an alternative and somewhat puzzling view of concepts as the referents of predicate-expressions. For Frege it is perfectly possible that we should have lack of knowledge concerning the concepts to which the expressions we use 'refer'. (See e.g. 1891, p.158). Frege developed such a view, we shall suggest, on the basis of considerations which were quite alien to the theory of concepts (and of conception) as such; they belonged, rather, to the theory of number: Frege's peculiar view of concepts is a consequence of the position into which he was forced by his work in the philosophy of arithmetic. This position had other not easily tenable philosophical consequences, for example the view that functions (including, of course, concepts in Frege's sense) must be defined for every object in the universe, and that it was therefore acceptable to conceive determinate totalities of objects, all objects, and of all functions in the universe. Thus we may reasonably demand of Frege's theory of number that it have some substantial advan-

tages of its own if it is to outweigh the doubtfulness of consequences such as these.

At first sight it seems that Frege may indeed go far toward meeting this demand for he is able to develop an ingenious analysis of number-statements ('there are 52 cards in the deck', 'there are 1,000 leaves on the tree', there are 50,000,000 Englishmen alive'...) according to which

the content of a statement of number is an assertion about a concept (Gl, § 46).

A Note on Herbart, Riemann and Frege:

As Frege acknowledges on p.III of Gl, it was Herbart who put forward the first explicitly concept-centred theory of number. In Herbart's words

every number refers to a general concept of what is counted; but this concept can remain completely undetermined in that for the determination of number it is quite irrelevant what it is which one is counting. (1824/25, p.161)

To the number 12 we add in thought the concept of a stool, or of a thaler, such that one becomes aware that the determination of number annexes itself to the concept undividedly and all at once. (Loc.cit.)

Not only Frege but also the mathematician Riemann (whose early enthusiasm for philosophy had expressed itself in a virtually exclusive absorption in the work of Herbart) were influenced by Herbart's insight into the nature of ascriptions of number. (Cf. the first paragraphs of Part I (The Concept of an n-fold Extended Manifold) of Riemann's Probe-Vorlesung, 1854). The present author has suggested that it is possible to develop, on the basis of Riemann's work, a generalisation of the Herbartian analysis which would go beyond Frege's theory in being applicable, for example to continuous as well as to discrete manifolds. Such a generalised theory would give an analysis in terms of concepts not only of numbers but of all characteristics of manifolds generally, something which it is possible to do within Frege's theory only by means of a roundabout appeal to the 'arithmetisation' of analysis (and of manifold-theory in general). In this way it would be possible to remedy what is perhaps a defect

in Frege's theory of the applicability of mathematics that it gives a direct analysis of the applicability of elementary arithmetic (which yields immediately to Frege's methodology of one-one correspondence) but an indirect analysis - which would certainly be alien to the physicist - of the applicability of higher mathematics. In this respect it is interesting to note that not only Husserl, through Cantor (see Rosado Haddock, 1973, p.140, and Cantor, 1932, pp.477-78) were influenced by Riemann's theory of manifolds, but so also, of course, were Einstein, Clifford, et al.

Let us now return to Frege's assertion that the content of a statement of number is an assertion about a concept. What one discovers on reading Frege's account of this analysis is that the examples which he adduces to justify its significance are all examples which tacitly lean towards the traditional view of concepts as the meanings of general terms, meanings, that is, which have been established as epistemically accessible. Thus statements such as

(1)"All whales are mammals" (G1 § 47)

(2)"There exists a nonrectangular equilateral rectilinear triangle"(G1 §53)

(3)"All human beings are mortal"(RHu,p.327, Eng.trans.p.332).

are all put forward by Frege as statements about concepts in his sense, where only in one passage (the 2nd paragraph of G1, § 46) does he make clear the extent to which examples such as (1),(2) and (3) are non-typical in that they could serve equally as examples of concept-expressions in the traditional sense. This may have been because Frege wished to draw on the long tradition of the concepts as (general) senses view in order to make his own theory more easily understandable to the philosophical community. But it may also have been because Frege appreciated, at least tacitly, that when interpreted as an analysis in terms of the traditional reading of 'concept' Frege's theory presents itself as a theory of some novelty and importance. When the artificiality of Frege's own notion of concept is taken into account, this novelty and importance are seriously threatened however.

The suggestion that Frege deliberately preserved the possibility of an equivocal reading of 'concept' receives further support from the passage where Frege tells us that his view, that a statement of number expresses an assertion about a concept

is perhaps clearest with the number 0. If I say 'Venus has 0 moons', there simply does not exist any moon or agglomeration of moons for anything to be asserted of;

thus the number cannot be ascribed, as, for example, Husserl would wish, to any manifold or totality of objects (for of course here there is no such totality), rather

what happens is that a property is assigned to the concept moon of Venus, namely that of including nothing under it.

Frege's theory is indeed 'clearest with the number 0'. But this is in virtue of the fact that under the traditional view of concepts as the senses of general predicate expressions this is the only number⁸⁶ for which Frege's theory holds.

For when one identifies concepts as the senses of general predicates it becomes meaningless to suppose, for any given concept, that one can give any a posteriori or empirical delimitation of all the objects which fall under that concept in such a way that a number could be 'attached' to it. For even if it could be shown that one had assembled together all the earthly objects falling under a concept such as whale, nothing could guarantee that there are no whales on other planets, however distant, and it is not merely on medical grounds that one would wish to question the possibility of an adequate survey of all the whales in the universe. Given that it is impossible to give an a posteriori delimitation of all the objects falling under any particular concept (qua general predicative sense), it follows that such a delimitation can be made, if at all, only on a priori grounds. A little reflection reveals that this is possible, outside mathematics, only in cases of concepts which are a priori impossible to be instantiated, for example the concepts red and green all over, not identical with itself, wooden iron, and so on. That is to say, Frege's analysis can succeed, under the given interpretation, only

in cases of concepts which are to be ascribed the number 0.

The above remarks may serve to emphasise the extent to which Fregean 'concepts' have been distanced from the concepts of the tradition, for not only (i) has Frege developed an account of concepts as the referents of predicates, defined for all objects in the universe. He has also, in order to make the given account supportable at all, (ii) moved away from the traditional notion in allowing his 'concepts' to be particularised, that is to be determined by particular objects, times, places, of the extra-conceptual world. Not only 'moon', 'whale', 'cathedral', 'thoroughbred horse', etc., are seen as denoting concepts on the Fregean view, but so also are 'moon of Venus', 'whale on the planet Earth', 'thoroughbred horse owned by Kaiser Wilhelm II on his 25th birthday', etc. All the expressions which result when all possible substitutions of object-names in such expressions, are conceptual expressions in Frege's sense. And by the arguments above it follows that all number-determinations which we meet in our extra-mathematical experience - excluding those which concern a priori non-instantiable concepts - are to be interpreted in terms of such particularised 'concept'-expressions.⁸⁷

The most immediate argument against Frege's interpretation of a posteriori number-determinations as assertions about particularised concepts is that, of the two alternative readings:

(1) there are 1,000 objects falling under the concept whale in the Atlantic Ocean,

and (2) there are, in the Atlantic Ocean, 1,000 objects falling under the concept whale,

it is clear that it is (2) which is the more natural, a fact which lends support in turn to Husserl's view that it is the manifold or totality to which the number immediately applies. The preferability of (2) arises not out of empty stylistic considerations: it is a direct consequence of the naturalness of the traditional view of concepts as against the artificiality of the Fregean view. Frege's insistence that numbers apply only to concepts meant that he was unable to accept any reading of the form of (2) as logically correct.

Thus consider how Frege tackles the apparent objection to his theory of number

that a concept like inhabitant of Germany would then possess, in spite of there being no change in its defining characteristics, a property which varied from year to year, if statements of the number of inhabitants did really assert a property of it. In reply to this it is enough to point out that objects too can change their properties without their preventing us from recognising them as the same. In this case however we can actually give the explanation more precisely. The fact is that the concept inhabitant of Germany contains a time-reference as a variable element in it, or, to put it mathematically, is a function of the time. Instead of 'a is an inhabitant of Germany' we can say 'a inhabits Germany', and this refers to the current date at the time. Thus in the concept itself there is already something fluid. On the other hand, the number belonging to the concept inhabitant of Germany at New Year 1883, Berlin time is the same for all eternity. (Gl, § 46 - this passage is of such importance that we have had occasion to refer to it twice already, in note 28 and on p. 94 above).

What is noteworthy from the Ingardenian point of view about this passage (which bears witness to the depth of Frege's philosophical insight) is that it has something important to tell us even when interpreted in terms of the traditional view of concepts as meaning-entities. The presence of 'variables' in the content of concepts, and of meaning-entities in general, is a formal-ontological feature of such entities which serves to distinguish them absolutely from entities belonging to the category of objects, and which determines also the peculiarly compact, systematically hierarchical structure of the former category as against the 'contingent', non-compact, non-systematic structure of the latter. Frege, unlike, say, Meinong, recognised this incompatibility of structure between the two categories. But he thought that he had discovered - in the principle of extensionality - a means of taking the incompatibility into account in a way which would not disturb the harmony of his object/function dualism. Whether he was successful in this regard is something which we shall have to consider (in § 45) below.

§ 14. Husserl: Proper and symbolic presentations of totalities.

From the arguments of the previous section it will be clear that we have to deal with two conflicting (or at least outwardly conflicting) theories of number:

- (1) the Fregean theory, according to which number is something which is assigned to a concept and then only to the totality of objects (if any) which fall under the concept,
- (2) the Husserlian theory, (which has a long history, see Angelelli, 1967, 3.6 and Ch.10), according to which number applies first and foremost to the totality of objects numbered, and then only to the concept (if any) under which the given objects fall.

The opposition is less clear cut than may at first sight appear, of course, since the use of 'concept' in (1) and (2) is not identical. Unfortunately the theses expressed in (1) and (2) each fail to supply us with anything which would determine in any positive way the ontological status of numbers, in a way which could be used to fix more precisely the relation between the two theories. The immediate temptation is to interpret the Fregean analysis as involving a conception of numbers as the properties of concepts, and the Husserlian analysis as one which involves a conception of numbers as the properties of totalities.⁸⁸ Yet the former interpretation was explicitly repudiated by Frege (see the first quotation on p.124 below), for whom numbers are themselves objects (in the sense that they are, in Angelelli's terms 'ultimate subjects of predication', Angelelli, 1967, p.235). And the latter does not form part of the theory presented by Husserl in UBZ and PdA (although it may be consistent with that theory).

The orthodox conception of Husserl's theory of number as presented in UBZ and PdA is that it is a psychologistic theory of the type which was attacked by Frege in his review of PdA. We have already sketched in § 12 above the lines along which a repudiation of such a view would proceed: for Husserl, both in his programmatic descriptions of the task which he had set himself in PdA, and in the actual carrying out of these tasks, was concerned

not with the evidence for arithmetical statements, but rather with the clarification of the meaning of such statements in the light of their actualisation in our cognitive experiences of counting, manipulating the symbols of arithmetic, applying arithmetic to other areas of experience, and so on. The orthodoxy of the 'psychologistic' conception of this work has grown in tandem with the influence of Frege in whose review of PdA it was first advanced. Any examination of Husserl's

early writings in the philosophy of mathematics reveals [however] that Frege, far from directing a "crushing attack" (Føllesdal, 1958, p.9) upon Philosophie der Arithmetik, did not even understand the view of number which the book expresses. (Willard, 1974, p.97f; Cf. also Mohanty, 1974, and Rosado Haddock, 1973, Ch.VI).

It must be admitted that Frege's misunderstanding of PdA can be attributed, at least in part, to a certain terminological laxity on Husserl's part. This we may impute to an overzealous appreciation by Husserl of the task which he had set for himself of providing an epistemological foundation for arithmetic which would clarify the meaning of the statements of arithmetic in a way which would reveal their relation to first-person cognitive experience. This meant that Husserl had to find some way of relating numbers (characterised, broadly, as formal concepts - see note 84 above) to acts, not only acts of counting but to all species of acts in which number-determinations, determinations of magnitude, etc., are involved. Thus where Frege had been led to an analysis of the different forms of statements of number, Husserl turned instead to an analysis of number acts. More precisely he turned to the spectrum of actual and possible presentations (Vorstellungen) against the background of which numbers figure in our experience, picking out as specially important those presentations which are presentations of totalities. From the standpoint of Husserl's later phenomenology within which we are able to distinguish actual from potential intentions of totalities, expressed from non-expressed, fulfilled from non-fulfilled, clear from confused, attentional from horizontal intentions of totalities, etc., this is crude talk indeed. Yet there is some gain in ease of understanding (and in historical authenticity) if we persist in it, so long as we are careful to avoid the mistake of

supposing that 'Vorstellung' is to be understood, as in Frege's philosophy, as designating a necessarily incommunicable aspect, part or event in some individual person's mind (Willard, 1974, p.98, and RHu, passim). (It is also safe to assume that any analytically-minded philosopher who finds himself unable (or unwilling) to understand Husserl's talk of presentations will fare little better with Husserlian intentions.)

To see what precisely Husserl did mean by presentations - if the invitation to reflect upon the continuous passage of one's own thoughts, each of which involves the presentation of certain objects, events, states of affairs, etc., is insufficient - we shall do well to consider Frege's interpretation of what Husserl meant and to point out where Frege went wrong. Totalities are formed, according to Husserl, whenever a collection of objects are connected together in a presentation. Frege tells us that he has been unable to follow the account which Husserl gives of this 'collecting together' :

I must confess that I have been unsuccessful in my attempt to form a totality in accordance with the instructions of the author. In the case of collective connections, the contents are merely supposed to be thought or presented together, without any relation or connection whatever being presented between them.(PdA,p.79). I am unable to do this. I cannot simultaneously represent to myself redness, the Moon and Napoleon, without presenting these to myself as connected; e.g. the redness of a burning village against which stands out the figure of Napoleon, illuminated by the Moon on the right. (RHu, p.322, Eng. trans.p.329).

It is clear from this passage that Frege held Husserlian presentations to be, in effect, pictorial images. (Compare the note to p.69 of G1 where Frege uses the term "Vorstellung" 'in the sense of something like a picture'.)⁸⁹

The examples given by Husserl throughout PdA, and the theory of 'symbolic presentations' which is perhaps the major contribution of that work, reveal that nothing could be further from what Husserl meant by 'presentation' than 'pictorial image', although certain presentations will per accidens be associated with such images. A more adequate conception of Husserlian presentations has

already been intimated in Ch.1. and will form the basis of Chapters 5 and 6 below. This is to conceive presentations as generalised meaning-entities. On the lines of the proposal of Angelelli to interpret Frege's 'Bedeutung' as 'significance' or 'importance' (1967, 2.26), one could then express the nature of a presentation of a totality of objects not wholly metaphorically as the importance of that totality for the presenting subject. In this way also one can see the way toward reconciling the two alternative approaches to the nature of number-ascriptions (1) and (2) (on p.98 above), since totalities under which fall purely arbitrary objects will rarely have sufficient 'significance' to be 'collected together' and given a number. This interpretation is supported by Husserl's remark that

The same objects can be presented in different forms of totality. Instead of thinking of the objects as collectively bound together without our picking out any special trait [ohne jede Bevorzugung], we can, according to the direction of our interest, bring out this or this group and so in a more or less multifarious way build totalities of totalities. (PdA, 163f, my emphasis).

Husserl's theory then comes to have the advantage that 'presentation' (or 'significance') can be interpreted sufficiently widely to include purely arbitrary totalities in a way which Frege failed to recognise. For a presentation in this sense is constituted even on the basis of a bare running through of the names of the objects involved (say 'apple', 'carrot', 'typewriter', 'Napoleon', etc.), whether in overt utterance or in covert 'attentional glances', in the absence of any visual imagery being called forth in association with these names. (Cf. the arguments of §§ 4 and 5 above, and the text to note 8). Presentations may also be constituted 'at a distance' - where the individual objects involved in the totality presented are themselves given neither intuitively nor in this 'symbolic' fashion. Only the totality as a whole (the swarm of bees, the line of trees, the gaggle of geese, the inhabitants of Germany) being given. And clearly this implies that the totality itself may be given either intuitively - when I see a line of trees - or symbolically - when I read about the population of Germany.

Symbolic presentation is not contrasted by Husserl with intuitive presentation, i.e. with presentation associated, per accidens with perceptual or phenomenal data. It is contrasted rather with what Husserl calls 'proper' presentation, a notion which may also be usefully explicated in terms of our significance/importance analogy. For what Husserl recognised, in effect, was that the indispensable condition for a totality (or purported totality) to be such as to have significance for a particular subject would be that the subject should be capable, at least in principle, of gathering its objects together in such a way that each was given individually as having equal rights with the other members of that totality. (We can walk down a line of trees, for example, or interview each member of a gaggle of geese).

The theory of symbolic presentation was then developed in such a way that it would be possible to extend Husserl's epistemological foundation of arithmetic beyond the area of application to totalities which are sufficiently small, closely-packed and homogeneous to admit of this kind of 'proper' presentation.⁹⁰ By means of this theory Husserl saw that it would be possible to develop a uniform epistemological account of number whilst ensuring that the content of the resulting concept of number would not be affected by arbitrary/empirical matters of fact such as, for example, the 'medical' inability of human beings to count beyond 10^{10} .

In order to develop an adequate theory of symbolic presentation, however, Husserl was constrained to fix criteria for symbolic presentations taken individually which would ensure that whatever was true on the basis of such a presentation would also be acknowledged as true on the basis of a proper presentation of the totality involved, (even if in some cases such a proper presentation was in fact unattainable). He saw that

if a content is not given to us directly as being that which it is,
but only given indirectly,

- where 'direct' and 'indirect' givenness correspond to 'proper' and 'improper/symbolic' presentation, then provided it is so given

through signs which characterise it uniquely, we have, instead

of a proper presentation, a presentation which is symbolic.

Thus we have e.g. a proper presentation of the outer appearance of a house if we are actually observing it; a symbolic presentation if someone gives us the indirect characterisation: the house on the corner of such and such sides of such and such streets. Every description of an intuitive object has the tendency to replace the actual presentation of that object by a surrogate symbolic presentation. Unique determining characteristics mark out the object in such a fashion that it can be recognised if occasion arises, and thus all judgments which are established on the basis of the symbolic presentation can be carried over to the object itself. Thus symbolic presentation serves as a provisional or even-in cases where the proper object is inaccessible, - as enduring surrogate for the actual presentation. (PdA, 193f, my translation here as elsewhere).

Note that despite the conceptual confusion revealed by Husserl's use of 'proper object' where 'object' alone is appropriate, there is nothing in the above to suggest that presentations are peculiarly subjective in any sense other than that they must, in each and every case, be presentations on the part of some particular subject. Exactly the same applies, as we shall see in more detail (in Chs. 5 and 6) below, to other kinds of meaning-entities.

On the basis of an interpretation of Husserl's Vorstellung as meaning 'pictorial image', Frege would, indeed, have had a just complaint against Husserl's theory of number; and in particular Frege's own theory would have been proved preferable in having a wider applicability. This is because the Fregean notion of concept

has a power of collecting together far superior to the unifying power of synthetic apperception [as also to the unifying power of pictorial imagery]. By means of the latter it would not be possible to join the inhabitants of Germany together into a whole; but we can certainly bring them all under the concept inhabitant of Germany and number them. (Gl, § 48).

It ought to be clear by now, however, that the actual account of presentation put forward by Husserl was such that its 'power of collecting together' is at least as great as Frege's conceptual combining. Where Husserl's theory decisively gains the advantage over that of Frege lies in the fact that Husserl

is able to preserve the universal applicability of arithmetic in such a way that both the epistemological roots of this discipline (in acts of counting, et al) and the verifiable content of any given statement of number are preserved. For Frege's 'conceptual combining' can be extended to numberable totalities where no common conceptual feature can be discovered, e.g. the totality $\langle \text{Moon, Napoleon, redness} \rangle$, only by 'objectifying' his concepts. This involves detaching them from the level of cognitional meanings in such a way that completely arbitrary 'concepts' such as the concept is either the Moon, Napoleon or redness come to be accepted as the norm. As Husserl recognised, totalities are in general

completely arbitrary and optional. In the formation of concrete totalities there is in fact no limitation whatever upon what particular contents are to be included. Any object of representation, whether physical or psychical, abstract or concrete, whether given in sensation or in imagination, and so on, can be united into a totality with any, and with arbitrarily many other objects. E.g., a few particular trees; the sun, moon, Earth, and Mars; a feeling, an angel, the moon, Italy; and so on. ... The nature of the particular contents $\left[\text{the concept, if any, under which they fall} \right]$ makes no difference at all. (UBZ, 298f., as trans. by Willard, 1974, 102).

The modern analytic philosophical use of 'property' to denote sets of objects arbitrarily selected from a pre-determined domain, (which implies, e.g. that a domain of 100 objects has 2^{100} 'properties' associated with it - see Haenjaeger, 1962, Ch.II), may be traced directly back to the objectified notion of concept which Frege had been constrained to develop in order to ensure the requisite generality of his theory of number. Where Frege went wrong, the analysts argue, was in supposing that it was unnecessary to predetermine in each case where we wish to refer to properties or concepts the object-domain involved; Frege held of course that it would in each case be possible to employ the determinate totality: all objects in the universe. As we shall see, however, both Frege and the analysts were wrong on this point. For whilst, as Frege recognised, objects determine their own totalities (modes of being, regional ontologies, object-strata), still the totality of totalities is not amongst those totalities which are so determined.

§ 15. Numbers as superstructural meaning-forms.

The identification of Husserl's Vorstellungen as entities which belong to the category of meanings-in-general can now be employed to show how it would be possible to develop a conception of numbers as themselves members of this category. This will suggest a way in which Husserl's formal analysis of number which (as we saw in note 84) was already present in PdA and which was preserved and extended in both LU and FTL, can be related to his 'psychological' or better: 'cognitional' analyses of number (analyses 'according to origin') which are to be found in his early works. (See especially PdA Ch.IV and UBZ § 1). (The exercise will also not be wholly irrelevant to our project of developing a coherent 'Fregean' philosophy of logical objects in general, and thus also numbers in particular, as meaning-entities.)

We saw in chapter I above how linguistic meanings fall into highly complex hierarchies. The most abstract meanings - those containing, in Frege's (and Ingarden's) terms, the greatest number of variables (see pp.30f and 97 above) - appear at the topmost nodes and give way to a series of lower-level meanings of increasing 'concreteness'. In fact Husserlian presentations, and all other members of the category of meanings-in-general, exhibit just this kind of hierarchical arrangement. A most general and abstract presentation of, say, four (completely undetermined) objects, gives way to a presentation of, say, four apples, which in turn gives way to presentations of 3 ripe apples and 1 unripe apple, 3 large ripe apples and 1 small unripe apple, 3 large ripe apples which are mine and 1 small unripe apple which is yours, and so on.

There is clearly the temptation to identify the number four as the topmost node of such a hierarchy, and there is much in Husserl's PdA which would support such an identification. Thus, for example, in the conclusion of his argument against Frege's analysis of number-assertions Husserl writes:

...the expression "assertion about a number" normally refers to relations which the number bears to other numbers. But this is not

the only kind of assertion about the number which is possible. This too is an assertion about the number: that the unities which it numbers have certain determinations; for through such an assertion the number itself (and here we speak only of the general presentation and not of the number as abstractum) is determined in such a way as to become a concrete presentation. (PdA, p.166, my emphasis).

The parenthetical warning reveals that Husserl uses the term 'number' in (at least) two different senses throughout PdA. In one sense we are referring to the number as a (relatively) general presentation, and in a second sense to dem Abstraktum Zahl. Before discussing what Husserl might mean by 'number' in this second sense, let us note the effect, in reading PdA, of taking care always to interpret 'number' in whichever sense is appropriate for a given context. For what happens is that many otherwise baffling passages acquire an immediate clarity of meaning. For example:

The number owes its genesis to a certain psychic process, which joins together the objects to be counted - and in this sense these objects are the carriers of the number. (PdA, 163)

- a passage which Frege was able to exploit to support his charge that PdA was a work of 'psychologism' - can only mean:

The number qua general presentation owes its genesis to a certain psychic process, which, etc.

Similarly a statement such as:

The number does not say anything about the concept of the numbered, rather this concept states something about the number, (PdA, 166), which is cryptic indeed when 'number' is read as 'arithmetical number', for under this reading Husserl would seem to be implying that given, say, there are four horses pulling the emperor's coach, then

'There are four horses pulling the emperor's coach',

is a sentence which tells us something about the arithmetical object which is the number 4! What the concept does 'state something about' is the general presentation involved, which may even be identified simply as the meaning of the expression: 'four horses pulling the emperor's coach'.⁹¹

Thus we have established the possibility of a conception of 'concrete' numbers (cf. Angelelli, 1967, 10.51), and we must move now to consider possible 'abstract number' conceptions in the hope of arriving at an adequate philosophy of numbers qua mathematical entities. One possibility would be to regard abstract numbers as objects obtained by some form of object-generating process of abstraction. Such processes, which will be repeatedly before our attention in the chapters which follow, may most appropriately be conceived as means by which we can associate object-entities in some systematic way with the members of a pre-given totality of meaning-entities. But such associations only have interest when the object-entities involved are not themselves concrete existents whose ontological status is independent of the process involved (e.g. if we were to associate the mental act, considered as a physiological event, with the meaning actualised in the given act). The interest of 'abstraction' processes arises when the objects themselves are generated by the process in question (as, for example, literary characters are generated by the communal abstraction-process, on the part of author and readers, carried out on the basis of the complex meaning-entity which is a literary work.) We shall introduce the term 'referentialising abstraction' or 'referentialisation' to designate such generative abstraction-processes, but these terms should not suggest that these processes are in any sense alien to the tradition of philosophy: as we shall see they have been familiar, in different guises, at least since Cantor and Meinong.

In order to defend a conception of abstract numbers as mathematical objects, it would be necessary to define a process of referentialising abstraction carried out on the basis of particular concrete presentations of totalities. The problem here is that the most immediate candidate for the results of such a process would not be any unique, fully abstract arithmetical number as such, but the determinate totality or set of objects (if any) which are involved in each particular presentation. (If the latter is considered as an 'intensional' entity then the former may be characterised as its associated 'extension'). Such totalities would themselves be arranged in a partial

hierarchy which would reflect, to a limited degree, the structure of associated meaning-hierarchies. The reflection would be partial only for two reasons: (i) Hierarchies of presentations which are 'transparent', that is, are epistemologically rigid designators in the sense outlined in §§ 3 and 4, will be such that one totality is common to all the presentations which make up the hierarchy involved. (ii) Many presentations, and sometimes even whole hierarchies of presentations, have no corresponding totalities at all. This is the case, for example, if I present to myself four imaginary and indeterminate apples; and it will continue to be the case no matter how many concrete determinations I build into the presentations involved.

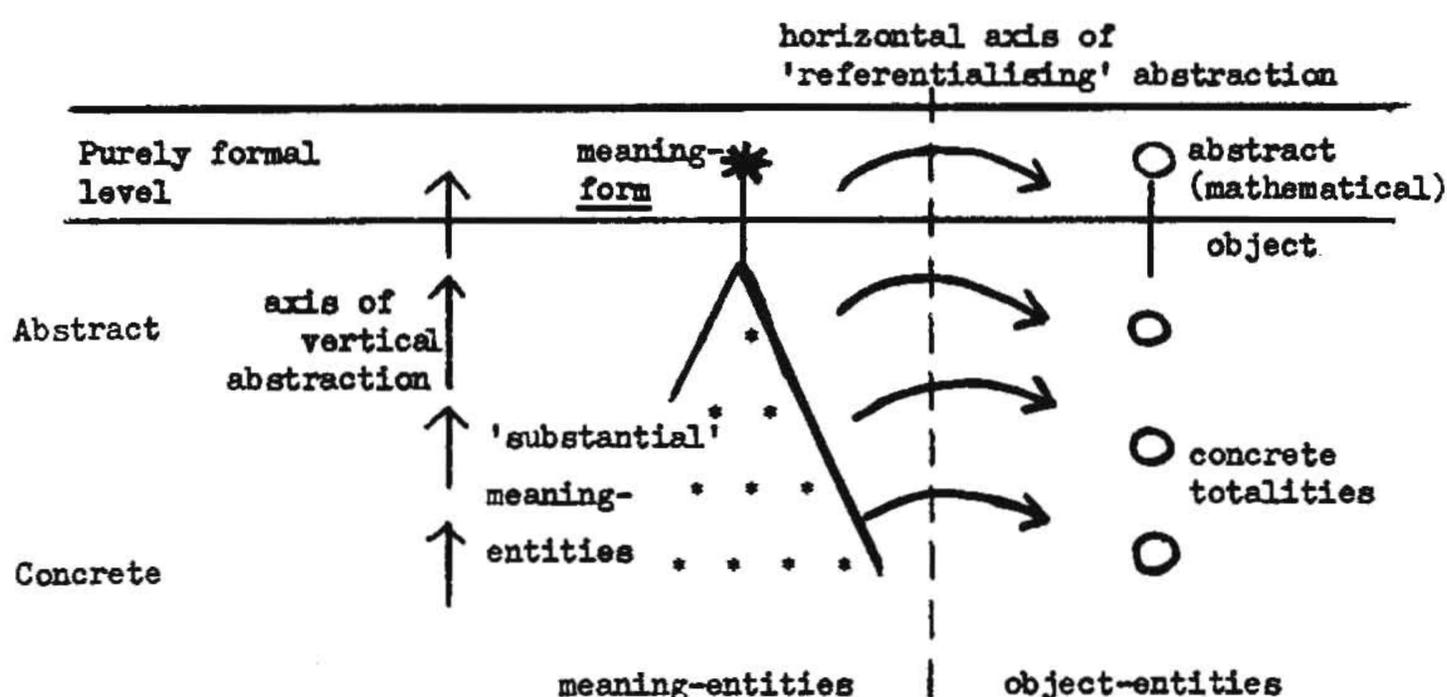
Thus we cannot immediately identify 'das Abstraktum Zahl' as an abstract object. One way to solve the problem of how we get from particular concrete presentations - with their associated totalities (if any) - to abstract entities which may serve as the subject-matter of arithmetic, is to recognise that there is, in addition to the 'horizontal' axis of referentialising abstraction, a second, 'vertical' axis of abstraction. Abstractions which take place along this second axis as applied to particular meaning-entities (e.g. to concrete presentations) yield not object-entities but rather new meaning-entities of a greater degree of abstractness to that which is possessed by the meaning-entities with which we begin. Thus relatively concrete presentations with a small number of 'variables' in their content, give way, when subjected to a process of 'vertical' abstraction, to relatively abstract presentations with a larger number of variables involved. Eventually by means of such a process a top-most node of a given hierarchy is reached, when as many as possible of those constants which are present in the content of lower-level meanings are transformed into variables in a way which is consistent with the given hierarchy. It must be stressed that the 'process' involved need not be an actually executed process; it need not even be conceived as something which could be systematically carried out by the actualising subject. For we are referring only to essential possibilities of transition from one position

to another in a given meaning-hierarchy. Similarly the process of referentialisation need refer only to the essential possibility of moving 'out' of a given meaning-hierarchy to the totality of objects of reference (if any) which are associated with a given node.

What we must recognise is that 'vertical' abstraction may be carried beyond the sphere of actualised or actualisable meanings-proper which constitute a given meaning-hierarchy (all of which are 'substantial' meaning-entities in the terminology of § 10 above). For with the aid of higher-order acts of the type by means of which we gain access to the superstructural level of meaning-forms, 'vertical' abstraction can be extended in such a way that it generates the meaning-form associated with any given hierarchy of substantial meanings. The suggestion now is that abstract numbers should be identified as the meaning-forms generated by the appropriate higher-order vertical abstraction-process carried out on the basis of corresponding concrete presentations of totalities. In this way 'abstract' numbers (numbers in the strict sense) are determined in such a way that their relationship to the particular concrete acts of counting, etc., is preserved, yet in such a way that they are isolated from the subjective 'impurities' associated with the concrete presentations which such acts involve. We can also begin to see a way in which arithmetic, the science of such abstract numbers may be properly conceived as a sub-discipline of logic, the science of superstructural meaning-forms in general. One problem which remains is the problem of the necessary truth of arithmetical statements something which, as we shall see, has been held to be irreconcilable with a conception of numbers as meaning-entities.

§ 16. Logic and mathematics

We have isolated two axes of abstraction, a 'horizontal' axis which yields objects or object-entities from determinate meaning-entities with which the former can be associated, and a 'vertical' axis along which we obtain relatively abstract meaning-entities from abstraction carried out on the basis of corresponding relatively concrete meanings or presentations. Along the latter we can obtain also - given the possibility of higher-order abstractional acts - the meaning-forms associated with the whole of a corresponding hierarchy of 'substantial' meaning-entities. The inter-relations between the two types of abstraction may be indicated by means of the following diagram:



It will be seen from this diagram the way in which the process of referentialising abstraction can be conceived as giving rise to an abstract mathematical object, e.g. in the cases so far discussed, to a number qua abstract object. What must be noted however is that, since there is in general more than one way of 'referentialising' given meaning-entities, including super-structural meaning-forms, it follows that the number qua object need not be unique. For every process of referentialising abstraction takes place against some 'background theory' which determines the nature of the results of that process as a function of the meaning-entities to which it is initially applied.

At the most trivial level such background theories are incipient in the process of learning the meanings of e.g. number words as names of objects. (Cf. Benacerraf, 1965, 47-62). At a higher level they are involved in the various possible 'reductions' of arithmetic (to logic or to set theory) which have been attempted, e.g. by Frege in Gg, by Russell in PM, and the now classical reductions of number theory to the theories of ZF and NBG sets.

Thus we can agree with Benacerraf when he argues that arithmetic

is not a science concerned with particular objects - the numbers. The search for which independently identifiable particulars the numbers really are (sets? Julius Caesar?) is a misguided one. (Op.cit. p.70).

But the truth of these remarks seems to imply the need for an increase in the number of ontological distinctions which we are called upon to make⁹², - it certainly does not justify the ontological miserliness evinced by Benacerraf, with his conclusion that

There are not two kinds of things, numbers and number words, but just one, the words themselves. (p.71).

Just as logic may be identified as the discipline whose subject-matter may be identified as the superstructural or 'purely formal' level of the realm of meanings, so mathematics may receive a parallel interpretation, that is as a discipline whose subject-matter is the structural or purely formal level of the domain of object-entities, both concrete (see p. 51f above), and abstract or intentional. Logic and mathematics thus come to be conceived as being related not as infant to adult ('Mathematics forms the manhood of logic' - Russell) but as brother to twinned sister. For as Husserl remarked

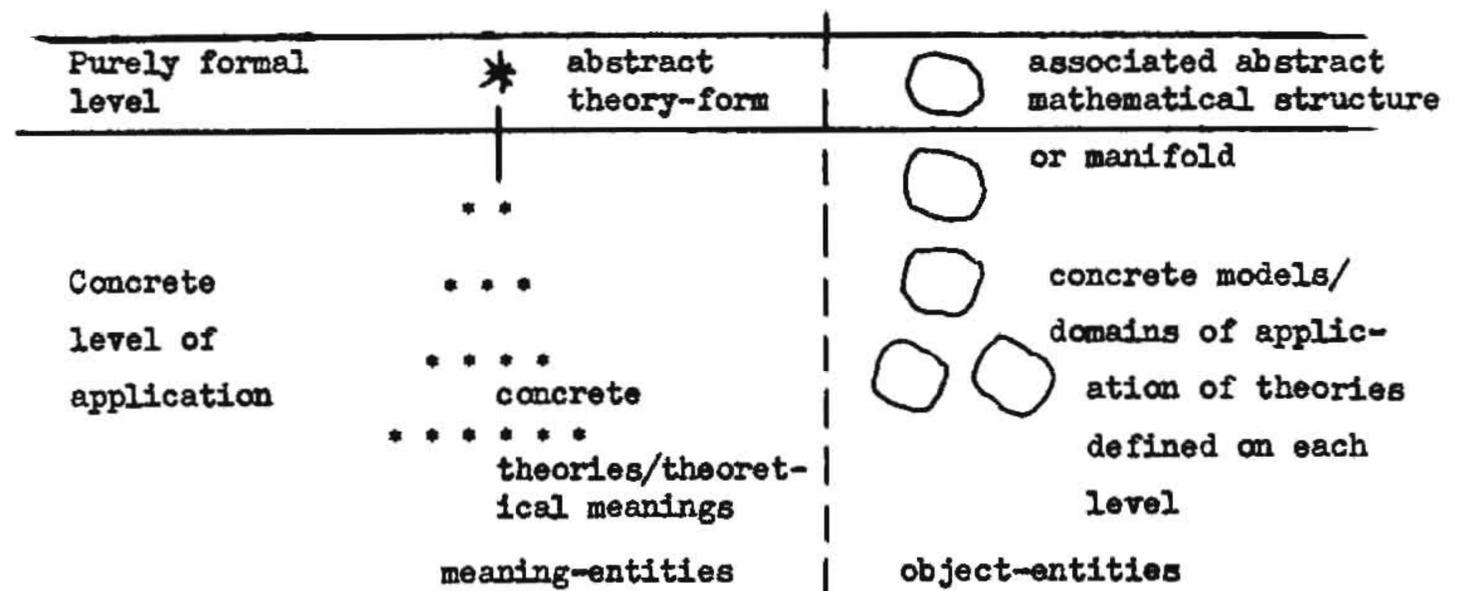
Everything that is logical falls under the two correlated categories of meaning and object (LU, II, 95; LI, 325).

In the present framework it also becomes possible to explain the direct applicability of higher-level mathematical disciplines, that is to say without need for recourse to any theoretical 'translation' of the disciplines involved into logical terms in such a way that the applicability of its 'minimal constituents' would be explainable along Fregean lines. (Cf. p. 93f.) For

associated with each theory is the theory-form of that theory, which bears identically the same relation to the theory itself as each lower-level meaning-form bears to the individual substantial meanings which instantiate it. The given theory-form determines, now, its 'objective correlate', the form of a possible field of knowledge over which a theory of the given theory-form will preside, (LU, 248f; LI, 241).

The latter is uniquely and solely determined by the fact that it falls under a theory of such a form, whose objects are e.g. such as to permit of certain forms of association, such as to be subject to certain laws of a determinate form, and so on. Thus the objects involved remain quite indefinite as regards their material constitution. Now just as all actual (substantial) theories are specialisations of corresponding theory-forms, so all 'fields of knowledge' are specialisations of corresponding manifold-forms. This is just to say that the domain of objects which forms the subject-matter of a given theory (or which can serve as a model for a given uninterpreted formal system) always consists of objects which satisfy the formal rules determined by the theory-form of the given theory and reflected in the manifold-form which is associated with the latter.

Again the situation may be conveyed by means of a diagram, parallel to the diagram above for individual meanings or presentations but constructed to apply to whole theories:



The 'theoretical meanings' involved in this meaning-hierarchy will be, in general, complex actualised meaning-entities which result from particular

actualisations or interiorisations of 'full' theories. The latter may be assumed to occur at higher levels in the hierarchy of theories and 'theoretical meanings' which is determined by a given theory-form as applied to a given subject-matter,⁹³ a hierarchy which will be, in general, infinitely complex.

We have deliberately left aside any positive specification of the nature of manifold-forms and of the elements (element-forms) out of which they are constituted. As we shall see in § 21 below, subtle problems arise when we raise the question of the object-status of these entities, especially in their role as forming (part of) the subject-matter of the positive discipline of mathematics. Traditionally it had been held to be indispensable to assume not only that mathematics possessed a world of pre-existent objects as its subject-matter, but even that such objects have an ideal (atemporal) existence. Only thus, it was claimed, would it be possible to explain the necessary truth of mathematical statements. In the section which follows we shall show, however, that the objects of mathematics are in an important sense irrelevant to the problem of necessary truth. In this way we hope to clear the way for the idea, which will be defended in the remainder of the work, that it is contingently existing (abstract) objects which form the subject-matter of mathematics.

§ 17. Platonism and Necessary Truth

The justice of our claim that numbers are 'primarily' meaning-entities may be seen to lie in the fact that the number qua object can be obtained only retrospectively, on the basis of referentialisation of the appropriate spectrum of substantial and formal presentations-of-totalities, the latter being themselves conceived as entities belonging to the category of meanings-in-general. But this seems to imply a dependence of numbers upon contingent matters of fact (e.g. the existence of presenting subjects, or of suitably-formed totalities of objects), something which would be in dangerous conflict with the fact that numerical statements may be logically necessary. Clearly the solution to this kind of difficulty can be achieved only by means of a clarification of the abstraction 'processes' which are involved, processes which, as we have suggested, need not be temporal processes except in a purely metaphorical sense. Indeed the notion of primacy to which we appeal in the initial sentence of this paragraph may be characterised as a logical rather than as a chronological notion. (Comparisons are in order here with the metaphorical employment of temporal adverbs in mathematical logic, not only by the intuitionists but also, for example in the exposition of Zermelo's cumulative type hierarchy, where one can assert the existence of a set consisting of all those objects satisfying a certain specified condition only if all of those objects are 'already' members of a pre-existing set. (See e.g. Fraenkel, Bar-Hillel, and Levy, 1973, p. 157).)

It may be thought that all such temporal metaphors could be eliminated, and with them all seemingly 'psychologistic' talk of abstraction and of abstraction-processes. We shall attempt to show however that not only are formulations of mathematical and other ontological issues in these terms logically harmless, they are even indispensable if we are to come to grips with the 'fine structure' (Frege, Gl, IV) which is possessed by even the simplest concepts of logic and arithmetic.

Difficulties which arise in connection with the necessary truth of

logical and arithmetical propositions of the type intimated above apply also, as has been pointed out by Moss (in A), to the interpretation of Frege's philosophical position on these issues which has been put forward by Dummett. 'Pure abstract objects', Dummett argues,

are no more than the reflections of certain linguistic expressions, expressions which behave, by simple formal criteria, in a manner analogous to proper names of objects, but whose sense cannot be represented as consisting in our capacity to identify objects as their bearers. (FPL,505).

But then, as Moss argues,

if an object exists as nothing more than the reflection of a linguistic usage, its existence cannot be necessary, since the existence of language is a contingent matter. (A, 2.1).

The closeness of the parallel between this objection and an equivalent objection which could be made against our own theory is revealed by the fact that the most abstract general presentations, which form the final staging-post before it becomes possible to move up into the purely formal level, can be obtained, we should argue, only against the background of an adequate numerical symbolism.⁹⁴ In this sense we too hold that numbers are 'reflections of certain linguistic expressions' even though, unlike Dummett, we believe that some such 'reflections' - which satisfy the highly specific criteria laid down in Ch.12 below - must be admitted as fully fledged objects.

As a first answer to this common difficulty perhaps I will be allowed the right to appeal to an argument put forward by Dummett, though in a completely different context, in an attempt to reconcile the following three theses concerning (Fregean) thoughts:

- (1) there are certain entities (Fregean thoughts) to which truth values are ascribed,
 - (2) these same entities are also identified as the senses of (complete) sentences (Ged, 60f, Klemke, ed., 511),
- and
- (3) 'it would still be true that the Earth has only one (natural) satellite even if there were no human being to express the fact or recognise its truth,' (FPL, 369).

The force of (3) is that whatever it is to which truth values are applied that 'application' is in a certain sense 'atemporal': the bearers of truth values do not 'acquire' those values at some determinate point in time, and they do not risk 'losing' them as a result of the passage of events. This seems to conflict with (1) and (2) however, since these imply that the existence of truth bearers, i.e. of thoughts (or propositions) is a temporal matter, that thoughts exist only given the existence of an appropriate language (and thus also of suitably trained users of the language, suitable contexts of 'serious' use, and so on). The orthodox solution to the difficulty is to reject (1) by invoking new entities (states of affairs or 'facts') whose existence is independent of any linguistic entities with which they would be only contingently associated (if at all). Frege recognised however that there was a way in which (1) could be used in support of (3), namely by developing a conception of thoughts themselves as timeless entities, whose status as meaning-entities is preserved in such a way that the need for any 'platonistic' commitment to timeless object-entities (facts, states of affairs) is avoided. As we shall see there is an important element of truth in Frege's conception, resting on the fact that the counterpart of platonism with respect to meaning-entities is, for important reasons, a more acceptable position than platonism with respect to objects (naive platonism). But such 'platonism of meanings' in general, and Frege's own approach in particular, leave us in the dark concerning the mode of actual cognitive access to 'thoughts' or 'propositions'. Such approaches imply also that the importance of thesis (2), some version of which is indispensable to any phenomenology of meaning, becomes relatively diminished with deleterious effects. (as we have seen in Ch.1) on Frege's theory of meaning.⁹⁵

Let us therefore return to the solution to the difficulty which is hinted at by Dummett, involving an 'atemporality' of the ascription of truth value to an entity. Dummett's argument rests on the possibility of a recategorisation of certain expressions, long held to be ineliminably referential (categorematic), as (syncategorematic) turns of speech:

In saying that, even if we were not there to express the thought that the Earth has only one natural satellite, that which makes the thought true would still be the case, we do not have to be construed as implying that there is any entity which is the referent of the expression 'that which makes the thought true': all that we are aiming to convey by means of this form of words is that it would still be the case that the Earth had only one natural satellite. (FPL, 370).

This linguistic tinkering yields a more substantial harvest, however. For given the (non-Fregean) conception of a thought as something whose existence depends upon the existence of a consciousness capable of grasping it, we are led to the view

that there are two conditions for the statement, 'The thought that the Earth has only one natural satellite is true', to be true: first, that the Earth should have only one natural satellite; and secondly, that there should be such a thing as the thought that the Earth has only one natural satellite, i.e. that there be a language in which that thought could be expressed or a being who was capable of grasping it. (Loc.cit.)⁹⁶

But this in no way implies that it is impossible, or even difficult to express the fact that only the first of these two conditions should be satisfied under certain given conditions - indeed thesis (3) above is a trivial instance of such expression. What Dummett recognises is that from the use of expressions such as (3) 'there need be no implication that there is an entity of any kind' whose existence somehow constitutes the fulfilment of that first condition. (Loc.cit.)

We can now exploit this argument - in a context where, as we shall see, it would be unavailable to Dummett - in order to support the generalised claim that there are two conditions for the truth of any statement concerning meaning-entities whether these be thoughts, individual word-meanings, numbers, or whole theories. Consider, for example, a statement such as 'There are two orange-trees growing on the side of Mount Etna'. This has as its truth conditions: first, that there should be two orange-trees growing on the side of Mount Etna, and secondly - to use a somewhat stilted Husserlian terminology - that there should be some being who has grasped or is capable of grasping the two orange-trees together as forming a numberable totality, as bearing to each other

the relation which Husserl calls 'collective combination' (Kollektive Verbindung) (UBZ, 301).

Again, this is not to say that there are not many occasions when it is right and proper to express the fact that for a given number-statement, and in particular for arithmetical statements of the form ' $2 + 2 = 4$ ', only the first of these two conditions should be satisfied. But the disclosure of these two sets of truth conditions does imply the consequence that no arithmetical statement is necessarily true in the strict sense of 'true' (i.e. in the sense of 'true' as meaning, in this context, 'satisfies both the given sets of truth conditions'). Thus there is a real basis for the anxiety expressed by Moss concerning the necessary truth of arithmetical statements under a phenomenological or Dummettian regime. What Moss fails to acknowledge, however, is that all purely arithmetical statements retain an analogue of necessary truth - which satisfies the logical requirements of mathematics - in the sense that they necessarily satisfy the first of the two given conditions.

Husserl's own early writings on arithmetic often seem to imply that he disallows the propriety of supposing that any statement of number could be such that only the first of the two given conditions could be satisfied: this is the philosophical content at the root of the impression that these works are works of 'psychologism' or 'anthropologism'. For the given position - if Husserl indeed held it - would tilt the balance between cognitional or epistemological considerations and strictly logical considerations regarding statements of number much too far in the direction of epistemology. The position seems to be implied, in particular, by Husserl's designation (both in UBZ and in PdA) of the relation of forming a numberable totality as a 'psychical relation'. Willard has convincingly demonstrated however that the 'psychological' implications of this term are restricted, for Husserl, to a quite different field of problems, having no bearing upon the problems of necessary truth considered above. Husserl was attempting to capture a distinction between two types of relations amongst objects of reference, that

is between

(i) relations which hold between objects only in virtue of something in these objects, the reciprocal nature of which would constitute the ontological support of the relation involved. (Cf. Ingarden, 1975). Husserl called relations of this type 'physical relations', amongst which we can include

identity, gradation (of degree), the relations within a continuum, the combination of properties in physical objects, and the species genus relation, along with all other relations "of the familiar sorts" (Willard, 1974, p.105, cf. UBZ, § 2, V),

and (ii) relations which are 'relations', at least from the point of view of Ockham's razor, only as a matter of terminological indulgence, since they rest upon no corresponding determinations on the side of the objects 'involved'. The paradigm here is provided by relations of the act-object type, 'intentional' relations such as the relation which holds between, say, Carnap and myself when I read his works or confuse him with his father, or that which 'holds' between Gladstone and Holmes as a consequence of the latter having been represented as the father of the former in an obscure Sherlockian coup de plume.

Relations of type (ii) are designated by Husserl as 'psychical relations'

amongst which fall, besides the act-object relation, also the difference which appears between two objects as seen through a prior judgment that they are different (UBZ, loc.cit.), along with the collective combination itself. As Husserl says, every content whatever, regardless of its nature, "...can be conceived of as different from, and also as collectively united with, every other. These two latter cases [of relations] are, precisely, cases where the relation does not immediately reside in the phenomena themselves, but, so to speak, is external to them." (UBZ, p.333) - Willard, 1974, p.105.

What Willard stresses is that only the first of the three kinds of 'psychical' relations distinguished above

is "mental" or "subjective" in the ordinary sense. The latter two are called psychical only because the manner in which they comprise

their terms is the same in kind as that found in the act-object relation, which specific relation was, for Brentano, characteristic of the subjective or the mental in the common sense.(1974,p.105).

Perhaps the matter can be expressed as follows: We feel unhappy about designating some types of 'psychical relation' as relations because the fact that given object-entities are 'related' by them is a fact which resides not in the objects themselves but in the presentations which we have of them. This suggests that such relations are not members of the category of objects-in-general (as are, e.g.'physical' relations), but that they are rather relational concepts, that is to say, entities belonging to the category of meanings. It follows that propositions about such 'relations' have two truth conditions of the types isolated above. Suppose for example that R is the relation forms a numberable totality with. Then the first condition for the truth of, say, 'Napoleon, the Moon and redness stand in the relation R' would be that Napoleon, the Moon and redness do stand in the given relation, and the second condition: that there should be some being capable of grasping that the given objects do stand in R, i.e. that they do form a numberable totality. Here however the first condition is trivially satisfied in all cases, since the power of number is such that all object-entities whatsoever can be brought together (clearly not necessarily in any spatial sense) and counted. A 'collapsed' first truth condition is possessed also by statements about the relational concept is different from, and we suggest that Husserl was feeling his way toward this notion of a collapsed first truth condition in his indication that these two just discussed cases are quite special even amongst 'psychical' relations in general (UBZ,p.33).

Possessing such double truth conditions is, we said, something which is peculiar to all statements concerning meaning-entities, and it is the possibility of independent satisfaction of the first condition in each case which renders a platonism of meaning-entities less objectionable than a platonism of objects: for the meaning-platonist has at his disposal the argument that by a given turn of speech, which seems to involve commitment to platonistically existing entities, what he really means is the coherence of supposing indep-

endent satisfaction of this first truth condition.

Interestingly these considerations are reflected in a purely formal argument, put forward originally by Anselm in his De Veritate, revived by Bolzano (for references see Berg, 1962, p.61) and more recently by Moss, (A, § 4), which purports to prove, by means of a reductio ad absurdum argument (unavailable when we are dealing with object-entities) that there exists at least one proposition.

If, with some trepidation, we introduce variables x, y, z to range over all entities generally (and not merely over object-entities), then we can introduce a predicate 'prop' which is to hold of all and only those entities which are propositions. We can also introduce (with more trepidation) predicates 'T' and 'F' which are assigned to those entities which are true and false (which are the bearers of truth and falsehood) respectively. Then we may reasonably assert

$$(1) \quad \text{prop}(x) \leftrightarrow Tx \vee Fx.$$

We now assume that

$$(2) \quad \neg \exists x \text{prop}(x)$$

i.e. that no proposition exists. That is to say, we assume the truth of (2),

i.e. $T(\neg \exists x \text{prop}(x))$, whence we can move to the weaker proposition:

$$(3) \quad T(\neg \exists x \text{prop}(x)) \vee F(\neg \exists x \text{prop}(x)), \text{ but then by (1)}$$

$$(4) \quad \text{prop}(\neg \exists x \text{prop}(x)), \text{ whence } \exists y \text{prop}(y) \text{ which contradicts (2).}$$

It follows that (2) is false, i.e. that $\exists x \text{prop}(x)$.

The mistake in the proof (assuming that a platonism of meaning-entities is in fact false) lies in the final sentence. For having assumed that (2) is true, it follows that we have assumed that both truth conditions for (2) are satisfied, namely that there exists a being capable of recognising the truth of (2), of formulating (2), etc., and also that (2). To prove the falsehood of (2) is therefore to prove that one or other (or of course both) of these conditions fails to be satisfied. That is to say we have proved either the negation of (2) or that there is no being capable of recognising its truth, etc. Either there are no conscious beings in the world, or there are propositions, a conclusion which is far from sufficient to establish the meaning-platonist's case by means of logical arguments alone.