1. Spatial Representation and Reasoning in Artificial Intelligence

Laure Vieu

1.1. Introduction

Space, like time, is one of the most fundamental categories of human cognition. It structures all our activities and relationships with the external world. It also structures many of our reasoning capabilities: it serves as the basis for many metaphors, including temporal, and gave rise to mathematics itself, geometry being the first formal system known.

However, space is inherently more complex than time, because it is multidimensional. In addition, even if unarguably less abstract than time, space is epistemologically multiple. Without going seriously into psychological studies, it is obvious that a larger number of knowledge sources (as vision, touch, hearing and kinaesthesis) contribute to establish mental representations of space. The multiplicity of spatial knowledge is also made evident by language. Space is grammaticalized in very few natural languages, and spatial concepts are spread over a wide range of syntactic classes: nouns (such as part nouns), prepositions, verbs (position and motion), adverbs, adjectives (shape).

Even though modeling spatial knowledge is clearly a crucial domain of artificial intelligence (AI), the difficulty of reducing all of it to a small number of primitive spatial concepts in an obvious or widely admitted fashion may explain why spatial representation and reasoning developed as an homogeneous clearly identified branch of AI much later than temporal representation and reasoning.

The earliest work appeared within other domains of AI and computer science, with a variety of objectives. Areas like robotics, physical reasoning, computer vision, natural language understanding, geographic information systems, and computer-aided design have contributed to the study of representing and reasoning with spatial knowledge, but space has not been really focused on as such there. A view widely held was that the ontology of space was unproblematic, topology and Euclidean geometry being the only mathematical models considered. This work then concentrated more on reasoning methods than on spatial representation, and, being closely linked to a variety of tasks, no real generic class of spatial problems and solutions emerged. It was even claimed that such a class does not exist (see Section 1.3.5). Only some of this work will be reported in this chapter, mostly in Section 1.4.

It is only during the last five years that fundamental studies on spatial representation and reasoning in AI, as such, have appeared in a significant number. A community has gathered now, as manifested by the recent increase in meetings and conferences (totally or in a great part) focused on the topic

(Mark and Frank, 1991; Frank et al., 1992; Aurnague et al., 1993; Frank and Campari, 1993; Guarino and Poli, 1993; Eschenbach et al., 1994; Amsili et al., 1995; Frank and Kuhn, 1995), and the series of workshops on spatial and temporal reasoning held in conjunction with major AI conferences such as IJCAI, AAAI, and ECAI since 1993. The main issues of the field have now begun to be identified, and a (still small) number of survey or field-defining papers and books have appeared (Davis, 1990; Freksa, 1991; McDermott, 1992a; Freksa and Röhrig, 1993; Hernández, 1994).

Although the term *spatial representation and reasoning in AI* still covers work toward the development of numerical algorithmic methods based on quantitative representations of space, the scope of this chapter is restricted to the symbolic approach, and it considers only qualitative spatial representations and symbolic reasoning methods. As a result, work more or less related to the field of computational geometry is completely left out here. For a survey of the field of computational geometry, the reader may refer to, for example, Preparata and Shamos (1985).

What remains in the scope of this chapter is the field that is now usually called *qualitative spatial reasoning*. This name was derived from the older field of qualitative reasoning, although at least two important differences should be stressed. First, the fields of qualitative reasoning and qualitative physics have in a great part dealt with scales discretizing quantitative domains or "quantity spaces" — that is, imprecise numerical values. Second, the representational problems raised by such quantity spaces being few, qualitative reasoning has then focused on reasoning methods. On the other hand, and however named, the field of qualitative spatial reasoning has given rise essentially to representational problems, which are far from being restricted to the discretization of a dense numerical space.

In the search for adequate spatial representations supporting genuine qualitative spatial reasoning, researchers in AI have started to give up the assumption that the ontology of space is straightforward and to address the issue of the nature of basic spatial concepts and spatial entities. Introducing several aspects of this ontological issue will constitute the core of this chapter. In the next two sections, some basic aspects of the ontology of space are discussed, and a number of approaches illustrating possible choices are presented. This presentation cannot thus pretend to cover exhaustively the work done in the field.

The subfield of spatial reasoning proper is then the subject of Section 1.4. Only a sketchy general picture of this field is given, considering the different ways of handling spatial representations as structuring factor. The very specific area of automatic geometric theorem proving is not examined there, even though algebraic geometry may in a sense be seen as symbolic and theorem proving techniques also. The interested reader may refer to Chou (1988) for a review of this area.

Finally, some issues that constitute interesting lines of current and future research in this field are emphasized in the last section.

1.2. Ontologies of Space and Spatial Knowledge Representation

1.2.1. GENERAL ASPECTS OF AN ONTOLOGY OF SPACE

The issue of the nature of spatial knowledge is certainly not a new topic raised by AI. On the contrary, it is an extremely old domain of research, investigated first in philosophy and mathematics, later in physics and psychology.

1.2.1.1. *Which Space?*

What is actually called *space* in all these disciplines varies significantly. It may be the real extent of the several dimensions in which we live. Physics postulates that we can observe and measure this physical space and then model it accordingly. It may be a cognitive representation of the physical space — that is, either the space human beings conceive from what they perceive, the space they store in memory and are able to recall through mental imagery, or the space they talk about. Modeling cognitive, mental, or commonsense space (or spaces) is here also an empirical enterprise, taken up by philosophy, psychology, and linguistics. Lastly, it may be an abstract construct, belonging to the class of mathematical structures that initially were built with the purpose of modeling the previous kinds of spaces.

Researchers in AI have intended to model either physical (for example, for robotics applications) or commonsense (for natural language processing purposes) space. In work aimed at giving general theories of space, this choice is sometimes left implicit. Moreover, arguments taken from both domains are often mixed. Confusion possibly arises from the fact that it is a widespread methodology in AI to rely on commonsense views for modeling physical as well as mental space. It might also be a sequel of the much held view that, even if the choice of primitives for an adequate representational framework is considered as debatable, the ontology of its *intended model* is still assumed to be unproblematic. For these researchers who perhaps adopt a Kantian view, space *is* Euclidean geometry without question. Reconsidering the classical distinction of goals in AI between obtaining efficient systems or cognitively accurate systems amounts here to distinguish clearly between physical space and mental space; and for both cases, we believe it useful not to take for granted that the ontology of space is a settled question. It is also plausible that in focusing on the differences between physical and commonsense space, one could find their commonalities and thus make further progress in the search for most general theories of space. Looking carefully at the properties of linguistic space with an AI perspective is, in these two respects, very important for the field. The chapter by Herskovits in this book (Chapter 6) contributes to this goal.

Whatever space is considered, the choice of a representational formalism of space and of its associated ontology is also based on several assumptions. The range of possible hypotheses is certainly not a novelty introduced by AI. These hypotheses recurrently appeared in the age-old philosophical debate on the nature of space.

1.2.1.2. *Absolute or Relative Space?*

The first choice to be made is between absolute and relative[1] space alternative made famous by the Newton-Leibniz controversy. An absolute space is a void, or "container," existing *a priori*, independently of the (physical or mental) objects that happen to be located in it. It is made up of purely spatial entities. In order to be used, such a space needs to be equipped with a location (or place) function giving each object its place. For its part, the relative approach denies the necessity of assuming the existence of an abstract independent space. A relative space is a construct induced by spatial relations over non-purely spatial entities — material bodies in the case of physical space, mental entities with more properties than just spatial ones in the case of cognitive space. The absolute option has been particularly fruitful in physics and is of course what mathematics deals with. The relative option seems to fit better with the cognitive approach, since the elements of an abstract space are not perceptible as such, but only through the existence of some material entities being located in it. It is also ontologically more parsimonious. Theories of relative space have, however, been much less developed than absolute ones.

If we assume the existence of an absolute space, we need next to choose between a space with a global structure or with only local properties. In what is here called a *global space*, each spatial entity is a location in a general reference frame so that its relative position with respect to all other spatial entities is already completely determined. An everyday example of a global space is a sheet of paper that serves as the backdrop for diagrams. In a *local space*, a spatial entity is situated through a number of explicit spatial relations with some, but not necessarily all, other spatial entities. The other spatial relations may be obtained by deduction, but some may be totally unspecified. An everyday example of local space is linguistic space, in which an object can be said to be close to another, without specifying in which direction. Thus, a global space is necessarily complete, whereas a local space can be either complete or incomplete.

This second question is closely related to the psychological issue of whether mental space is basically visual and constituted of images or depictions on which spatial relations between spatial entities have to be read (pictorial or analog or depictive representation) or whether mental space has a more linguistic flavor and is structured by propositions in which spatial relations relate spatial entities (propositional or descriptive representation) (Kosslyn, 1980; Paivio, 1986; Pylyshyn, 1985). A current position held is that (at least) both representational systems are present in the mind and involved in mental imagery processes (Kosslyn, 1980) or that, in fact, spatial cognition handles more complex structures covering both aspects (Tversky, 1993a).

Mathematically and computationally speaking, it amounts to choosing between a coordinate space, such as Cartesian geometry, or an axiomatic theory of space, such as Euclidean geometry, as in Hilbert (1971). In the former, spatial properties are implicitly given by the algebraic properties of numerical orders on several coordinate axes. Analytical geometry thus commits one from the start to standard analysis, as well as to a higher-order language, since a geometric figure is defined as a set of coordinate points. In the latter, usually called elementary geometry, axioms state fundamental properties of space. It does not commit to the use of number theory or set theory: congruence, determining distance, and order are not necessarily numerical; lines and planes are not necessarily sets of points.[2] Of course, there are strong links between the two kinds of structures. Cartesian geometry is a model of Hilbert's axiomatic system. But their underlying ontologies are very different, as are the reasoning methods applied to them.[3]

Since relative spaces are by definition theories of spatial relations, there is a strong similarity between relative spaces and local absolute spaces. At least two important aspects make the difference, however. First, they hold different constraints on possible basic entities. In a relative space, basic entities are constrained by the nature of the domain, so that their spatial properties depend on their other essential properties. For instance, in case of a physical relative space, the material constitution of physical bodies imply that they are all spatially extended. An absolute space is independent, so there is no other constraint than elegance, expressive power, and computational efficiency in the choice of basic entities. In a sense then, absolute spaces, in principle, are more general. However, to serve as an adequate representational framework, absolute spaces sometimes reflect some of the specific properties of physical or commonsense space. Second, their capacity to express motion is radically different. In an absolute space, the motion of a physical body is a change in value of the location function. In a relative space, the motion of a physical body is a change in spatial relationship with respect to some other bodies. Motion is then necessarily relative, which raises the question of the choice of adequate reference objects. More important, an absolute space is assumed to persist through time, so that expressing the continuity of motion amounts to stating that the location function is continuous. In contrast, a relative space is different at each different (spatial) state of affairs, so that continuity of motion depends on the identity of physical bodies through time, which is metaphysically problematic.[4]

For these reasons, or just because they were already more widespread in the literature, most authors in AI have chosen absolute spaces as representational frameworks. Classical mathematical global spaces (coordinate spaces) have been successfully used in computational geometry. In the knowledge representation area of AI, the necessity to cope with imprecision, incompleteness, and uncertainty of knowledge, both in physical space and cognitive space, led to drop this approach for a more qualitative one. As we will see, a number of authors still adopt a space that can be characterized as global. They keep the global axes for orientation and relax at least one of the undesirable characteristics of classical coordinate spaces, usually their metric, changing dense numerical orders for symbolic orders or discretizing them to deal with imprecision. However, local absolute spaces best fit these requirements. Being symbolic, they are particularly adequate for coping with imprecision and representing incomplete information. Gobal spaces require dealing either with disjunctions for the values of the location function or with numerical equations with free variables as in the algebraic geometry approach.

As a consequence, in the qualitative spatial reasoning community the kind of space adopted is generally local absolute space. This chapter focuses on local absolute spaces and describes only a few approaches based on global spaces.

Formal frameworks widely used for representing local spaces are of two types:

- Axiomatic theories, generally, first-order. These theories consist of a language (relation and function primitives) and a number of axioms.
- Relation algebras or "calculi." An exhaustive set of mutually exclusive primitive relations is given. The inferential behavior of these relations is given in transitivity tables, in the spirit of Allen (1983). These tables implement relation combination — the algebra of relations, a relation in the general sense being any disjunction of primitive relations.

Axiomatic theories are far richer in their expressive power. Indeed, only a small subclass of first-order axiomatic theories can be converted into relational algebras. On the other hand, reasoning in relation algebras, especially those based on binary relations, is made much easier by their good computational properties. Some authors use both versions of the same theory: they first present and discuss an axiomatic theory, more expressive, and then use an equivalent relational algebra for the implementation. It must be noted that, ideally, in both cases, a proof of which structures are the models of the theory should be given to fully characterize what the primitive concepts and entities are actually able to describe. Models, however, have been rarely worked out in the qualitative spatial reasoning literature.

The concrete elements of an ontology of absolute space actually are the basic spatial entities constituting the space, as well as the primitive spatial notions expressed over these entities. These two elements are actually interdependent, some notions being difficult, if not impossible, to express over some kinds of entities. We next introduce the kinds of basic entities and primitive notions that have been used for representing space. In the following section, we present several spatial representation frameworks developed in AI, classifying them according to these ontological criteria — first to the nature of spatial entities chosen as primitives, and then to the notions that are expressed.

1.2.2. ELEMENTS OF SPATIAL ONTOLOGIES

1.2.2.1. *Basic Entities*

As for time, for which there is an alternative between instant-based and period-based ontologies (van Benthem, 1983), basic spatial entities constituting absolute spaces can be of two types. On the one hand, space can be made of abstract lower-dimensional entities like points of classical geometry. On the other hand, basic entities can be extended portions of the same dimension than the whole space.

Choosing the first type of entities eases the development of theories because geometry has been investigated for a long time. Even though the objectives of AI are not those of mathematics, it helps to rely on well-known properties of the structures usually sought as intended models. Approaches based on points are presented in Section 1.3.1.

The second option does not benefit as much from mathematical results. The development of these theories is accordingly less advanced. Nevertheless, advantages are many, and much work is currently being done in this area. Extended entities are spatially more similar to the basic entities of physical or mental relative spaces. In particular, the argument of adequacy is often held by researchers aiming at modeling commensense (absolute) space. On the efficiency side, because extended entities are directly appropriate to serve as values of the location function, theories need not go higher order (for example, with sets of points as values) or be combined with an abstraction process (such as considering all objects as punctual). It is also possible for a theory based on extended entities to have a finite and still connected domain (space can be discrete without presenting gaps), thus facilitating their implementation.

The global-local option has an influence over the choice of basic entities. Point-based space can indifferently be local or global. To be accurate, in a global space points are no longer the real basic entities; these are the coordinates. Similarly, in a global space based on extended entities, the real basic entities are segments of the axes. This means that the shape of extended entities in a global space is fixed (in a Cartesian frame, rectangular or parallelipipedic), while in a local space, regions of any shape can be chosen. For extended basic entities, we then distinguish approaches where space is global and based on arrays of cells (Section 1.3.2) or tuples of intervals (Section 1.3.3), from approaches where space is local and based on regions (Section 1.3.4).

As for time, there are translation procedures between point-based ontologies and region-based ontologies. Defining one in terms of the other, for example, regions as sets of points and points as ultrafilters of regions,[5] requires the use of a higher-order language and thus is computationally unattractive. For this reason, when the knowledge to be represented bears on both kinds of entities, mixed ontologies are preferred. The domain of basic entities is then split into categories and the language includes incidence relations between them. The multidimensionality of space marks really the difference with time when more than two categories are considered, such as points, regions, lines and surfaces. As we already said, this is the case in some axiomatic theories of Euclidean geometry. Approaches based on mixed ontologies are presented in Section 1.3.5.

1.2.2.2. *Primitive Notions*

Spatial relations and properties are generally grouped into three domains: topology, orientation, and distance. Psychological studies have proved that these notions are acquired by the child successively, in this order (Piaget and Inhelder, 1948). We now give a quick overview of these three groups of concepts. In this description, we heavily rely on mathematical concepts, principally introduced in different families of theories: topology, metric spaces, Euclidean geometry, and Cartesian geometry, which are what we have in mind when we talk about classical mathematical spaces. It must be noted that the orientation and distance groups are not clearly distinguished in these mathematical spaces. In Euclidean geometry, orientation and distance concepts are intimately related. From the standard axiomatics of Hilbert (1971), the axiom groups of incidence, order, and parallels deal only with orientation; but the other two groups, congruence and continuity, involve both orientation and distance. In metric spaces — which include the most standard mathematical spaces, as $\mathbb{R}^3$ — the real-valued distance function induces both an associated topology and orientation. Even if they are not so clearly marked out in mathematics, in AI it has proved to be fruitful to investigate topology, orientation, and distance separately, sometimes for different applicative purposes.

Of course, there are important spatial relations and properties that belong to none of these groups. These concepts are significantly more complex and accordingly much less treated in the literature. This is the case of shape properties, or morphology, which will be almost left out of the discussion because no systematic account has been proposed up to now in a qualitative manner. However, some morphological concepts have begun to be introduced on regions (see Section 1.3.4.3).

Topology. Topological theories are generally seen as the most abstract spatial structures, the weakest geometries. Mathematically speaking, a topological space is a structure $\langle X, \Omega \rangle$ where:

- X is a set (the points of the space)
- Ω is a subset of 2^X (the *open* sets, or the topology)
- Ω includes $\emptyset$ and X
- the intersection of any two open sets (two elements of Ω) is an open set (belongs to Ω)
- the union of any number of open sets is an open set.

A *closed* set is the complement in X of an open set.

There are several ways of looking at the spatial notions brought about by topology (Hocking and Young, 1961). Under one perspective, topology principally defines the notions of boundary and contact. An open subset of the topological space is seen as including none of its boundaries and a closed set as including all of them.[6] The *boundary* of a subset $x \subset X$, noted ∂x, is then defined as being the difference between its *closure* (the smallest closed set including it, noted $\overline{x}$) and its *interior* (the biggest open set included in it, noted $\mathring{x}$). The relation of contact or *external connection* between two subsets can be based on the sharing of boundaries. An interesting derived property is the one of being a connected subset — a one-piece space portion.

According to another view, topology essentially is the theory defining the notion of continuity. A function between two topologies is continuous if it maps an open set onto an open set. Since it characterizes invariants under continuous deformations, topology has been called the geometry of the rubber sheet. Algebraic topology defines the rank of the homology group of a subset — that is, the number of its "holes" or discontinuities in its boundary. It thus gives a somewhat rough notion of shape distinguishing a doughnut from a ball.

In representing spatial relations in other frameworks than classical topology (in particular in spaces not constituted of points as basic entities), some of these notions sometimes take a quite different meaning. For instance, in region-based spaces, contact may be modeled without calling for the notion of boundary or even that of open set. Nonetheless, theories axiomatizing contact may still be called topological because they retain most of the classical properties of contact.

In classical topology topological notions apply to subsets and imply a notion of extension[7] — contrarily to orientation or distance notions that classically apply to isolated points. As a consequence, in AI the set-theoretical notions of inclusion, overlapping, intersection, and union are often grouped with the topology cluster, even though strictly speaking these are not topological notions. What is more, they do not necessarily suppose the membership relation of set theory. When modeled directly on extended entities, they constitute what is called a *mereology* — a theory of part-whole relation (Lesniewski, 1931; Simons, 1987). Accordingly, theories modeling topological concepts without set theory, taking extended entities as basic entities, are known as *mereotopologies*.

Orientation. We may distinguish two levels of basic orientational notions stemming from geometry. Relations of the first level are elementary in the sense that they enable the definition of relations of the second level, but in AI most authors treat orientation from the perspective of the second level only, introducing directly a complex system of relations.

First are the concepts of elementary geometry related to the notion of straight line (sometimes called arrangement): alignment between three points or incidence of a point on a line, betweenness of one point with respect to two other points and order on a line, congruence and comparisons of angles (pairs of lines or triplets of points), parallelism and orthogonality between lines. Betweenness is the primitive relation axiomatized both in Hilbert (1971) and Tarski (1959) as the basis of orientation. These concepts of elementary geometry are not easily transposed to ontologies that are not based on points.

Second, are the concepts of vectorial geometry enabling an axis (a directed line or *direction*) to establish an order throughout the space, not just on a line. This order presupposes implicitly the notion of orthogonal projection on an axis. Vectorial geometry is not restricted to point-based spaces. In global spaces, this orientational process is given a particular prominence, since the entities themselves depend ontologically on a *reference frame*, usually composed of several axes as in Cartesian geometry. Actually, reference frames are often used in local spaces as well, only implicitly through the use of binary relations. Two or three orthogonal axes (for two- or three-dimensional space) are frequent, thus yielding relations such as *is right of*, *is front-left of*, or *is Northwest of*, depending on the labeling of the reference frame axes. But more generally, a system of any number of axes dividing the space into several sectors or cones may be conceived.

In local spaces, considering orientation relations generated by a reference frame raises the problem of choosing a specific reference frame together with a labeling. Which reference frames are used? When are they appropriate? Even though it is not possible to answer them on purely spatial grounds, these questions have been debated to some extent, maybe because of their great importance in human communication, reference frames being often left implicit. These and other aspects of linguistic space are discussed in this book in Herskovits's chapter (Chapter 6). Here, let us just recall that literature distinguishes between three types of orientation:

- *Absolute orientation.* A unique, more or less arbitrary, reference frame is used. The two-dimensional reference frame of cardinal directions ⟨north/south, east/west⟩ is common in representing geographic space. The inherent reference frame of a global space yields, of course, an absolute orientation.
- *Intrinsic orientation.* The intrinsic reference frame of the reference entity (the second argument in a binary relation like *is left of*) is used. Intrinsic reference frames exist only for extended entities and originate in a variety of inherent properties of the entity: shape (particularly symmetry), motion, typical position, functional properties etc. The most common intrinsic reference frames in three dimensions consist of three axes: up/down, front/back, and left/right.
- *Contextual, extrinsic and deictic orientation.* The intrinsic reference frame of another entity, contextually salient but distinct from the reference entity, is used. When this entity is the speaker or the observer, we have a case of deictic orientation. When it is neither the speaker, nor any of the arguments of the spatial relation, orientation is called extrinsic.

Distance. In elementary geometry, the notion of *relative distance* is introduced through the relation of congruence between segments — that is, through a quaternary relation of equidistance on points: *x and y are as far apart as are z and t* (Hilbert, 1971; Tarski, 1959). Relative distance can also be introduced symbolically with the ternary relation *x is closer to y than it is to z.* This is the kind of distance most easily expressed in local spaces, whether point-based or region-based. However, what is usually called distance is the numerical function on which metric spaces are defined. More precisely, a *metric distance* is a function d mapping pairs of spatial points onto $\mathbb{R}_+$ such that $d(x,x) = 0$, $d(x,y) = d(y,x)$, and $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality). It is worth noting that a metric space is not necessarily Euclidean. For instance, curved spaces (where parallels meet) admit of other distances.

As already mentioned, in geometry there is an inherent link between distance and orientation. Classical global spaces, having dense numerical orders on the axes, enable the definition of distance functions (Euclidean or not — that is, preserving congruence or not) that makes them metric. On the other hand, because metric distance is properly additive, it induces orientation:

$$Between(y,x,z) \equiv d(x,z) = d(x,y) + d(y,z)$$

Similarly, betweenness could, in principle, be defined in terms of the relative distance relation *is closer to* on points, exploiting an equivalent of triangle inequality (van Benthem, 1983):

$$Between(y,x,z) \equiv \forall t(t = y \vee closer(y,x,t) \vee closer(y,z,t))$$

In qualitative representations of space, a metric distance is rarely used, and, when a valued distance function is sought, it is replaced by a *discrete distance.* In discrete spaces like occupancy arrays, distance is simply a function on $\mathbb{N}_+$. In these cases, the dependency between distance and orientation is altered, and, for instance, "circles" may become squares. Of a more qualitative flavor are theories modeling concepts such as *far* or *close* or, in fact, any qualitative scale discretizing the continuous real domain. These discrete distances are sometimes called *naming distances.* Like for any other "quantity space" of qualitative physics (Bobrow, 1984), general purpose theories accounting of fuzziness and granularity phenomena may be used. However, it is in general difficult to axiomatize properly the additivity of distance and triangle inequality, thus loosing essential spatial properties and the link between distance and orientation.

Even though they are most often modeled on point-based spaces, distance concepts can be transposed to spaces based on other kinds of entities, in a rather straightforward fashion.

1.2.3. ADDITIONAL FEATURES

After choosing the ontology of a space, there are still a number of other possible general assumptions on its nature. Any of these parameters may be left unspecified, giving a more general but incomplete theory, which is computationally a drawback.

Space can be assumed to be bounded or unbounded (for any pair of points there is always a third one situated further in the same direction, or there is no maximal region), discrete or dense (between any two points there is a third one, or between any two nested open regions there is a third one), or even continuous (betweenness on points satisfies the Dedekind-cut property), and, if basic entities are extended, atomic or dissective (any region has a proper part).

All of these parameters affect of course the finite nature of the domain of the resulting theory and thus the computational properties of its implementations. For this reason, sometimes a difference is made between the intended ontology of space and the space which is actually represented. For instance, space can be assumed to be dense but *represented* as being discrete (Habel, 1994). In this case, the choice of a granularity is not intrinsic but dependent of a specific task. To really capture the "density in intension" of the intended models, the capacity of integrating various granularities in the same theory (switching granularities or combining them) is then an important further parameter.

Finally, the dimensionality of space may be fixed (and is necessarily so in global spaces), whether two-dimensional, three-dimensional, or even four-dimensional if space-time is considered.

Still other aspects can be considered if one takes into account the kind of entities one wishes to locate in space. For instance, the distinction between *table-top* space, *small-scale* space, and *large-scale* space is often made in AI and in cognitive sciences in general. Some spatial notions are more or less relevant depending on these perspectives. More important perhaps, there are ontological dependencies between properties of located objects and properties of spatial entities that can be exploited. These aspects are more deeply discussed in this book in Casati and Varzi's chapter (ontological dependencies between the materiality of objects and space, Chapter 3) and in Frank's chapter (ontological constraints on the spatial properties of geographical objects, Chapter 5).

1.3. Overview of Approaches to Spatial Representation in AI

This section presents a number of well-known or representative approaches to spatial representation that take points, cells in arrays, tuples of intervals, and regions, as basic spatial entities. It then examines approaches based on mixed ontologies and finally briefly discusses the relationship between and time.

1.3.1. POINT-BASED SPACES

Approaches to spatial representation based on points and defining spaces typically focus on orientation and distance concepts, since d with topology would require going higher order.

1.3.1.1. *Orientation*

Elementary orientation relations applied to points are at least terna was noted before, the usual relation is *between*, which yields the alig of the three points plus an ordering. Such relations have been rarely u AI. Betweenness is axiomatized as a primitive relation in Vieu (1993) the lines of van Benthem (1983), and also in Borgo et al. (1996a) foll Tarski (1972). In these two papers, however, points are not really the entities of the theories; they are introduced as sets of regions or (impl as centers of spheres.

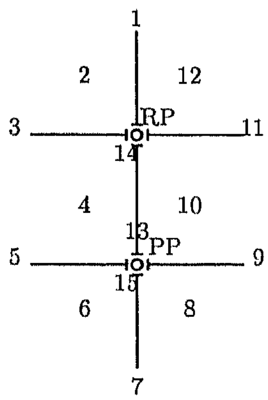
Representation of orientation concepts in AI is most often tackled the vectorial geometry point of view, which has been widely restric two-dimensional local spaces.[8]

Local reference frames. Hernández (1994), Freksa (1992b), and L (1993) all express the contextual orientation of a located point with re to a reference point, as seen from a perspective point. They apply t reference point a local reference frame in which the frontal direction is by the direction ⟨*perspective point, reference point*⟩. These authors al the relation algebra approach and describe the inferential behavior o primitive relations in transitivity tables.

Their approaches differ according to the orientational structu the reference frame. Hernández uses four axes for two-dimensional s resulting in the segmentation of the plane around the reference poin eight cone-shaped sectors, each centered around an half-axis: front, front, right, right-back, back, left-back, left, and left-front, yielding possible relations over the triplet.

Freksa uses only two axes, which segment the plane into four quad falling between, not around, the axes: right-front, right-back, left-back left-front. In addition, Freksa distinguishes the four half-axes as po positions: front, right, back, left. This also yields eight relations, but h quite different from Hernández's, since in this approach exact align between the three points may be represented. Exchanging the ro the reference and perspective points yields a new reference frame, an opposed frontal direction and applied now to the original persp point. Combining the two reference frames thus obtained, Freksa gets three axes making a double cross distinguishing six areas of the plane, six half lines, and the segment between the two points — that is, thirteen different relations, rising to fifteen if we add the two distinguished positions of the reference and the perspective points (Figure 1). Ligozat (1993) generalizes this calculus with any number of axes.

Figure 1. Freksa's fifteen relations



Order on angles. Orientation relations can also be modeled with circular orders — that is, with angle comparisons in the plane. In the most basic of these orders (Schlieder, 1995), one compares the relative orientation of, again, a located point with respect to a perspective point and a reference point, through the characteristics of the angle A = ⟨*reference point, perspective point, located point*⟩. The located point is then situated either on the direction ⟨*perspective point, reference point*⟩ itself ($A = 0$),[9] counterclockwise ($0 < A < \pi$) from it, or clockwise ($-\pi < A < 0$). This is nothing more than the, say, left-right orientation induced by a reference frame with a single axis. One can add the further distinction that the angle is either acute ($|A| < \pi/2$) or obtuse, corresponding then to a reference frame with two axes, distinguishing the four quadrants left-front, left-back, right-back, and right-front (Latecki and Röhrig, 1993).

Schlieder (1993) generalizes the approach. Going around the perspective point in a predefined sense (say, counterclockwise, the positive sense), starting from the reference point (or the direction ⟨*perspective point, reference point*⟩), any number of located points (or directions ⟨*perspective point, located point*⟩) may be ordered. This order correspond to the comparison of several angles of the form ⟨*reference point, perspective point,* *located point*⟩. The orientational information contained in these generalize circular orders is not equivalent to the straightforward use of a loca reference frame.

A representation system equivalent to circular orders is achieved by a axiomatic theory of directions in the plane (Aurnague, 1995). This theor takes directions as primitive entities, but those can be seen as equivalenc classes of pairs of points, considering pairs of the kind ⟨*perspective point located point*⟩. A ternary primitive relation $Kd(D1, D2, D3)$ is axiomatize so as to express that $D1$ is "closer" to $D2$ than to $D3$ — that is, tha the absolute value of the angle $\langle D1, D2\rangle$ is smaller than that of $\langle D1, D3\rangle$ Definitions of equal, opposite, and orthogonal direction to a direction an of median direction between two directions are introduced.

These last two approaches, based on angle comparison, are mor general and more qualitative than those based on local reference frame or on distinguished angle values. Indeed, in the latter two, the number o qualitative distinctions that the theory is able to make is fixed a priori What is more, these distinctions induce discontinuities in the space or the axes of the reference frame. In Hernández (1994), it is assumed that a located point cannot fall exactly on the lines dividing the sectors, thus excluding some positions, whereas in Freksa (1992b) the same status is given to a half axis and to full quadrant, which amounts to give infinitely more precision to some positions than others. On the other hand, in generalized circular orders as well as in the theory of directions, the number of angles that can be distinguished is not fixed and rather depends on the number of located points to be compared. In other words, the precision of the theory adapts itself to the knowledge to be represented, and all points of the space are treated homogeneously. If the importance of the upper, frontal, and lateral axes in cognition justifies the use of a priori axes in a particular application, then it is easy to add some particular points or directions corresponding to these axes in the representation. To be fully general, though, these calculi still require to be extended to three-dimensional or even n-dimensional space.

1.3.1.2. *Integrating Orientation and Distance*

Qualitative distance notions have been significantly less treated than orientation, and they are usually considered together with orientation notions. According to the terminology introduced earlier, we distinguish between approaches based on relative and discrete distance.

Relative distance. It is surprising that very little work has been done modifying axiomatic Euclidean geometry, dropping some possibly undesirable properties such as continuity (Archimedes' axiom). In Vieu (1993) introduced above, betweenness for orientation is complemented by the relation of ternary relative distance relation *is closer to*. The quaternary equidistance relation may then be defined in the same theory, but the integration of orientation and distance requires further improvement.

Zimmermann (1993) adds to Freksa's orientation representation system a linear order on edges or line segments (point pairs) in order to compare their length implicitly. The division of the plane is successively refined, taking first into account the order between the three segments generated by the three points (located, reference, and perspective points), then adding the three segments generated by the orthogonal projections of the located point onto the three axes of the double cross, and lastly enriching the order along the lines of the so-called Δ-calculus (Zimmermann, 1995). It is claimed that the integration of orientation and distance is achieved. However, it is not clear exactly what the resulting system is. The relational algebra combining all distinctions is not fully described in these papers. The mapping between distance and orientation constraints — although introduced at the representation level of the relative position of the three located, reference, and perspective points — seems not to be exploited in the transitivity tables.

Discrete distance. In Frank (1992a) and in Hernández et al. (1995), qualitative "naming" distances are introduced. Both articles deal with granularity scales from the simple binary near-far scale to a totally ordered scale of n distance steps or ranges, partitioning $\mathbb{R}_+$ into n intervals. Addition (or subtraction) tables compute the distance between a and c ($d(a,c)$) from $d(a,b)$ and $d(b,c)$, where a, b, and c are aligned.[10]. Hernández et al. (1995) considers a variety of restrictions on the distance scales, such as monotonically increasing interval length along the scale, and thus obtains several composition tables. Distance addition in the general case can only be approximated by a disjunction of values. But even if triangle inequality is respected, its limit case, equality, cannot imply the collinearity of the three points. This is so because in a discrete scale, extreme values may absorb the others — for instance, $d(a,b)$ = *very close*, $d(b,c)$ = *very close* and $d(a,c)$ = *very close* are possible simultaneously, whether the three points are aligned or not. As a consequence, in these systems, full integration of distance and orientation is by definition impossible to reach.

In Hernández et al. (1995) the conditions when a particular granularity, or scale, has to be chosen are briefly discussed. Parameters such as the nature of the orientational reference frame and the size of the reference object are considered. However, the problem of how to switch granularities according to a change in perspective or how to combine information at various granularities is left aside in both approaches, while it is what is really at stake when modeling granularity.

logical frameworks. This may explain why the work covered by the term diagrammatic reasoning is actually very diverse — for instance, see Glasgow et al. (1995). In addition, it is not clear what exactly remains of the pictorial space, if metric aspects are dropped, as the authors intend. Surely, topological and orientation information remains, but it seems obvious that areas such as language and vision integration and diagrammatic reasoning should deal with some morphological aspects as well.

As a knowledge representational tool for space, the most important problem this approach encounters is that of any global space — namely, the difficulty to represent partial or vague information. The problem occurs with topology, which is dependent on orientation (for instance, to represent the fact that Spain and France share a boundary, it is necessary to know their relative orientation), as well as with orientation alone (for instance, it is impossible to store the information that Holland is north of France without also knowing that it is north of Belgium and east of Great Britain).

The representational expressiveness as well as cognitive adequacy and computational efficiency of this approach has been extensively commented in Narayanan (1993).

1.3.3. INTERVAL-BASED SPACES

After the great impact of Allen's interval calculus (Allen, 1983) in AI (see Chapter 7), there have been several (similar) intents of extending it to a multidimensional domain (Güsgen, 1989; Mukerjee, 1989; Mukerjee and Joe, 1990). In this approach, spatial regions are represented by tuples of intervals that are the projections of the regions on the axes of a given absolute reference frame. Spatial relations between rectangular regions are then expressed by a tuple of relations between two intervals for each axis, in the relation algebra formalism. Since the relations belong to the set of Allen's thirteen relations (*before, meets, overlaps, starts, started-by, during, contains, equals, finishes, finished-by, overlapped-by, met-by, after*), only topological and orientational information is accounted for.

Even though intervals are situated relationally with respect to each other, one can consider that space is more global than local. Indeed, using intervals as coordinates, vectorial orientation is given the priority, and topology depends on it. But mereotopology is more thoroughly accounted for than in symbolic arrays. Direct representation of region overlap is possible, and one distinguishes between inclusion in the interior (*during*) and inclusion at the border (*starts* or *finishes*). In a way, this approach extends the symbolic array one in dropping at the same time discreteness and any remaining metric aspect. Relational, possibly partial, orders along the axes replace numerical orders. However, since aggregation has not been considered, all regions are rectangular (or parallelepipedic) shaped. As in arrays, they are all aligned along the axes of an arbitrary reference frame.

For those applications domain meeting these restrictions, this theory has the advantages of Allen's calculus: topology and orientation information is freed from any distance and length information, and reasoning is eased by the existence of efficient algorithms and transitivity tables.

The extension to nonaligned rectangles has been considered in Mukerjee and Joe (1990) for two-dimensional space. Space is then local, as each rectangle creates its own local intrinsic reference frame based on the knowledge of which side is the "front" of the rectangle. The ease of reasoning, however, is lost because there are more possible relations between two rectangles that are nevertheless less precise, so that the transitivity table is bigger and presents more ambiguity. There is an important additional drawback at the expressivity level. The angle between the frontal axes of the two rectangles is determined with an inaccuracy up to $\pi/2$ degrees, which yields a high imprecision. Further, topological information is completely lost. The same relation between two rectangles may describe situations in which one rectangle includes, overlaps, touches, or is disjoint from the other. The usefulness of this extension is then really dubious.

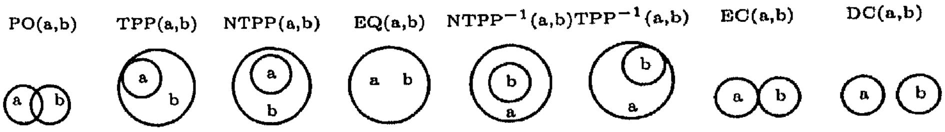
1.3.4. REGION-BASED SPACES

Contrarily to tuples of intervals, regions are extended entities of any shape. This property makes the representational systems based on regions much closer to spatial cognition, as we already said. In addition, they constitute the only first-order theories that deal properly with topological concepts. For these reasons, region-based theories and calculi can be viewed as the right extension of Allen's calculus for dimensions higher than one, regions being to material objects and space portions such as "the inside of the glass" or holes, what intervals (or periods) are to events and states. Actually, the notion of region is independent from dimensionality, *extended* entity meaning only having a nonempty interior — that is, having the same dimension as the whole space. However, the word *region* is commonly used for spaces of dimension higher than one only.

Most existing axiomatic theories stem from the work of Whitehead and Clarke on mereotopologies in formal ontology (Whitehead, 1929; Clarke, 1981; Clarke, 1985). This area of research has regained attention lately (a good survey is present in Varzi (1994; 1996a)) and has become quite developed in AI as well, as evidenced by the chapters by Casati and Varzi (Chapter 3), Cohn, Bennett, Gooday and Gotts (Chapter 4), and Galton (Chapter 10) in this book.

Whitehead and Clarke's theories were based on a unique mereotopological primitive relation C (connection), axiomatized as symmetric, reflexive, and extensional. Some authors now choose to separate mereology from topology with two primitive relations, such as proper part and contact or external connection (Borgo et al., 1996a). Still some others introduce boundaries in their domain, thus obtaining mixed ontologies (see Section 1.3.5 below). In Clarke's theory as well as in most others, it is possible to isolate an exhaustive set of eight mutually exclusive mereotopological relations (DC, EC, PO, TPP, NTPP, TPP-1, NTPP-1, EQ) and thus transform the axiomatic theory into a relation algebra (Randell and Cohn, 1989). See Figure 2.

Figure 2. RCC eight relations



For the mereological notions, Boolean operators of union, intersection, and complement are added, sometimes based on a general fusion operator (Clarke, 1981) as in standard mereology. For topological concepts on regions, two different approaches have been taken.

1.3.4.1. *Topology on Regions with and without Boundaries*

The first approach fully exploits the classical topological distinctions of open and closed regions. The introduction of interior and closure operators, in a way similar to that of Boolean operators, is made possible in theories where the complement definition and the axiomatics on C (or EC) are such that a region and its complement are not connected. This is the case in earlier theories such as Whitehead's and Clarke's.

Contrary to what Clarke seems to have thought, these theories do not capture just any standard topology restricted to extended regions — that is, without any region having an empty interior. Asher and Vieu (1995) have given the models of the theory presented in Clarke (1981). These are topologies restricted to nonempty *regular* regions[12]. However, as shown independently by several authors (Biacino and Gerla, 1991; Vieu, 1991), this theory, when extended with a certain definition of points as sets of regions (Clarke, 1985) (thus going higher order), has the unexpected result of having the same models as standard mereology (EC becomes an empty predicate). But it is still possible to give a more complex definition of points that preserves topological properties (Asher and Vieu, 1995).

With the topological notions of open and closed regions, different notions of contact may be defined: external connection between two closed regions or joint, contact between a closed region and an open one (as between a region and its complement), and weak contact or adjacency in which two closed regions touch without being connected. Contrary to the first two kinds of contact, weak contact implies a discontinuity — that is, the sum of the two regions is not connected[13] (Asher and Vieu, 1995). It must be noted that weak contact has an extension only in nondense models.

The same distinctions are introduced in Fleck (1996), which does not present an axiomatic mereotopological theory on regions but directly mathematical structures. The author shows how to constrain a classical topology on $\mathbb{R}^n$ and extract the right subsets corresponding to regions. In this work, weak contact is modeled as holding between two open regions whose common boundary has been deleted from the underlying space. The two regions are thus both not connected and zero distance apart. Two important differences can be noted between these structures and the structures that are models of the theory presented in Asher and Vieu (1995). On the one hand, even with deleted points, the remaining space may still keep the density of $\mathbb{R}^n$. On the other hand, a weak contact implies not only that the sum of the two regions is disconnected but also that the whole space is disconnected because space has gaps where objects touch. Change in touching is then difficult to cope with, and this solution seems to be more relevant for a relative space approach than for an absolute one. However, if the relative approach is in fact taken, then the appropriateness of $\mathbb{R}^n$ with its classical topological and metric structure needs at least to be discussed. For their part, pure mereotopological theories do not constrain the dimensionality of space, neither its Euclidean character. They thus have models that are classical topologies on $\mathbb{R}^n$, and models that are not (Lemon, 1996).

1.3.4.2. *Topology on Nondifferentiated Regions*

The second approach rejects the basic topological distinction of open and closed regions, on the grounds that there is no cognitive evidence of which regions corresponding to some location (for instance of physical objects) should be open and which closed. An additional concern regarding the theories of the previous approach is that, although they are not equal, there is no region that embodies the mereological difference between $\overline{x}$ and $\mathring{x}$ (the boundary). The best-known work in the approach based on nondifferentiated regions is the much developed axiomatic theory by Cohn's group in Leeds, the so-called RCC theory or calculus (Randell et al., 1992a; Gotts et al., 1996), although their first work belonged to the previous approach (Randell and Cohn, 1989). A special attention is given to computational issues, and the corresponding relation algebra is systematically derived. The chapter by Cohn, Bennett, Gooday and Gotts in this book (Chapter 4) thoroughly describes this work. RCC is also originally based on Clarke's theory, but a slight change in the definition of complement makes all regions (externally) connected to their complements. In the mereotopological part of the theory, the set of eight mutually exclusive relations described above is derived. The models are probably similar to those of Clarke's theory restricted to closed nonempty regular subsets, but no formal proof has been published.

Rejecting the open-closed distinction has a number of drawbacks. Different kinds of contact cannot be distinguished. More importantly perhaps, all models are then nonatomic, which is problematic on a computational point of view as well as on a cognitive one.

Quite similar results appear in Egenhofer and Franzosa (1991), taking the relation algebra perspective from the start. Observing that any subset x of a classical topological space can be represented with the pair of subsets $\langle \mathring{x}, \partial x \rangle$, the authors describe topological relations between two subsets through the emptiness or nonemptiness of the two by two intersections of their interiors and boundaries. This method defines sixteen relations between subsets that are reduced to eleven if the domain is restricted to (subsets corresponding to) closed extended regions and to nine if further restricted to regular regions — assumed in the paper to be what corresponds to relevant space portions of physical space. These nine are the eight relations described above, where partial overlap (PO) is split into two relations depending on the emptiness of the intersection of the boundaries of the two subsets. So, if the domain is further restricted to connected regions without holes,[14] the nine relations are reduced to the usual eight.

In the same approach that models contact without taking into account openness and closedness, there is the older work by Fleck on adjacency structures on "cells" and fuzzy spaces (Fleck, 1987). Cell structures differ from region structures in that, like in occupancy arrays, there is no mereological relations between cells — that is, no cell contains or overlaps another cell. Fleck shows that if adjacency sets (sets of cells sharing the same contact zone) are classified according to the dimension of the shared zone (point, line, surface, etc.), then adjacency structures over a cell partition of $\mathbb{R}^n$ fully characterize the topology of this space.

Actually, a recent trend tries to introduce distinctions between kinds of connections and to classify connectedness in mereotopological theories, whether with or without the open-closed distinction. The notions of "strong connection" (connection along a spatial entity of dimension $n-1$ in n-dimensional space) and "simple" region or "well-connected" region (one-piece region such that any two halves of it are strongly connected) are introduced in Gotts (1994b) and Borgo et al. (1996a). In this last paper, mereological notions are introduced with the part primitive (P), and *simple region* is the chosen primitive predicate to introduce topological ones — without the open-closed distinction, thus this theory clearly belongs to the second approach.

The definition of topological shape (connectivity order, which distinguishes between "doughnuts" and "balls") has proven to be possible on the sole primitive of connection (Gotts, 1994b; Gotts et al., 1996); this definition is also independent of the kind of approach chosen.

If region-based ontologies are clearly well suited to represent mereotopological information, they seem to be less well suited for orientation and distance information; or at least these notions on regions have been much less investigated up to now.

1.3.4.3. *Orientation*

The direct modeling of the basic arrangement relations such as alignment or betweenness on any kind of regions is not straightforward. The extension of regions makes it difficult to consider a unique direction of alignment between three regions and establish properties such as the transitivity of alignment.

However, orientation is involved in morphology, at least implicitly. First attempts to model morphology on regions can be found in two approaches. First, the convex-hull operator has been axiomatized in RCC. It enables a number of other concepts based on orientation and topology, such as "inside" (Randell and Cohn, 1989), and morphological aspects based on the position and number of concavities (Cohn, 1995) to be defined. In principle, betweenness on regions should be definable in terms of convex-hull as well. It remains to be proved that its desirable properties can be recovered.

In another approach, spheres[15] are characterized on the basis of a primitive relation of congruence between regions (Borgo et al., 1996a). In this work, basic orientation notions on regions are then axiomatized in two steps. Exploiting the correspondence between a sphere and a point (its center), the authors first transpose Hilbert's classical axiomatic geometry on points to spheres. Geometrical notions are then extended to any kind of region.

Working out the models of these two theories remains to be done. In addition, it is not clear that all the desired properties of either convex-hull or congruence are captured — that is, that these morphological notions on which orientation depend are properly axiomatized. Indeed, in Lemon (1996), it is shown that the axiomatization of convex-hull in the language of RCC must be dependent on the dimension of the space, which is not the case in RCC theory up to now.

The application of vectorial orientation concepts originally introduced on points has also been transposed to regions. Approaches choosing this option are in a way similar to the local version of multidimensional extension of Allen's calculus. Nevertheless, since orientation is there added to topology instead of being the sole primitive notion, the shape of regions is not a priori constrained, and topological relations are correctly treated with their own primitives. In these works, regions are assumed to be connected. In Hernández (1994), the full relative position of two two-dimensional regions is given by a pair of one of the standard eight mereotopological relations and one of the eight sector orientation relations (cf. Section 1.3.1.1).[16] The local reference frame yielding the orientation relations is modified to take into account the extension of the reference object but not that of the located object. And, as regards orientation, regions are supposed to be approximated by rectangles. Even though interesting deductions can be drawn when mixing topology and orientation, this approach is still quite restricted and cannot be easily extended to deal with morphology. In Aurnague (1995), the algebra of directions presented above (cf. Section 1.3.1.1) is combined with Clarke's calculus on regions. The relative orientation between two regions, along one direction, is given by a ternary version of Allen's relations, thus implicitly comparing the intervals that would result from a projection of the two regions on a straight line oriented by the direction.

1.3.4.4. *Distance*

Distance on regions has received even less attention than orientation, and in particular, it has not been considered in RCC yet. A pseudo-metric space on regions has been proposed in Gerla (1990), but distance is not qualitative in this work.

A relative distance concept between points may be adapted to regions and linked to mereotopology. The ternary relation *is closer to* would then compare the shortest distances between regions. The null distance between regions may be defined either as the smallest distance or as occurring between regions in contact. It has to be noted that difficulties appear when trying to adapt triangle inequality, because the extension of the intermediate region has to be taken into account. Thus the link between alignment and distance is not easily recovered. This may explain why this approach has not been explored up to now.

In Borgo et al. (1996a) the notion of sphere, together with mereotopological primitives and the morphological relation of congruence (on which sphere is also defined), gives in principle a powerful means to deal with distance. At present, only granularity aspects have been integrated in the theory (Borgo et al., 1996b).

1.3.5. SPACES WITH MIXED ONTOLOGIES

Some approaches are based on entities of various dimensionality. Their spatial ontologies mix extended and nonextended entities, without assuming interdefinability — for instance, without assuming that regions are sets of points or points sets of regions. Incidence relationship between entities of different categories replace ontological dependency.

Mereotopology on regions and lower-dimension entities. The motivations behind these approaches are twofold. On the one hand, some authors hold a cognitive justification before caring about ontological parsimony. They argue that humans unproblematically handle both real and abstract spatial entities in everyday spatial reasoning. In Gotts (1996b) and Galton (1996), any range of dimensions can be modeled. Gotts (1996b) chooses a unique asymmetrical primitive of incidence *includes a chunk of* holding between entities of possibly different dimensions, whereas Galton (1996) uses a part mereological primitive holding only between entities of the same dimension and a topological primitive *bounds*. Theories such as these should also in principle be able to define spatial concepts directly on the best-fit entities: topology on extended entities, and alignment and distance on points. It must be noted, though, that only mereotopological notions have been axiomatized in this approach up to now.

On the other hand, the ontological parsimony lost on the domain can actually be recovered on primitive concepts. Some authors are motivated by the ontological concern of having few, as general as possible, primitive notions. For instance, mereotopology can be axiomatized on regions and points with a mereological primitive only (Eschenbach and Heydrich, 1993; Eschenbach, 1994), assuming an additional primitive categorization (for example, with a *is a region* predicate). Another approach is taken in Smith (1993; ming). Smith axiomatizes mereotopology on regions and *boundaries* (entities of any lower dimension). His theory is based on two primitive relations, a mereological and a topological one, and no a priori categorization is made on the domain: *is a boundary* is a defined predicate. In these two theories, no attempt is made to model a range of dimensions of more than two categories.

Clementini and DiFelice (1995) mainly look for mathematical expressivity. Exploiting further the approach taken in Egenhofer and Franzosa (1991) (see Section 1.3.4.2), they consider the possible mereotopological relationships between two-dimensional regions and boundaries (lines) in classical point-set topology. Several representational methods are explored and compared, with set intersection, the three topological functions of interior, exterior, and boundary, and the dimension of the entities, as basic ontological primitives. The representation based on five primitive predicates (*touch, in, cross, overlap, disjoint*) and on two boundary functions is favored.

Geometry in the Qualitative Segmentation of Cartesian Space. An older tradition in AI assumed a global space, in general Cartesian space, as the underlying ontology for space. There, points (actually, coordinates) were the basic primitive entities. Within this ontology of space, higher-dimension entities such as regions are in principle defined as sets of points. However, in practice, to be able to reason qualitatively, the representations include several classes of entities. Some spatial relations, like topological ones, are introduced directly over regions and are axiomatized more or less independently of the fact that they are sets of points and more locally than globally (without relying on the Cartesian frame of reference). In these approaches, the intended ontology of space is point-based, but the ontology of the representation actually is a mixed one. Forbus's and Davis's important work aiming at qualitatively modeling space and motion can be seen from this point of view (Davis, 1988; Davis, 1990; Forbus, 1983; Forbus, 1995).

In Forbus (1983; 1995), the duality of this ontology is made explicit by using both a *metric diagram*, which is a bounded piece of two-dimensional Cartesian space ($\mathbb{R}^2$), and a *place vocabulary*, which is a local space with a mixed ontology of points, line segments, and regions. The place vocabulary segments the underlying metric diagram into space portions that are qualitatively meaningful with respect to both the static objects occupying space and the mechanical characteristics of the motion to be described. The assumption that space, whether physical or perceptual, is global, point-based, and essentially metric and that qualitative spatial representations are necessarily approximations with specific points of view of this underlying space, seems to have led the author together with other researchers to hold the so-called *poverty conjecture*: *there is no problem-independent purely qualitative representation of space* (Forbus et al., 1987; Forbus et al., 1991). Fortunately, now that a lot of work has been carried out in AI on representations of space based on other ontologies, as this chapter and this whole book should show, such a pessimistic statement has become outdated. However, it would be very interesting to investigate how the process of constructing the right place vocabulary from a particular metric diagram and for a particular task, can be analyzed. Constructing Voronoi diagrams seems to be a possible solution in some cases (Edwards, 1993). More generally, this problem must be related to the design task in spatial reasoning (see Section 1.4.3).

In Davis (1988; 1990), the separation between the underlying global space and its qualitative representation is less clear. The choice of classical

Cartesian space (here $\mathbb{R}^3$) as the underlying space is motivated by the fact that, up to now, no alternative theory to Euclidean geometry has been proposed that thoroughly and consistently models all aspects of space. In reasoning qualitatively about space, this work indeed relies on a high number of more or less well-known topological and geometrical theorems, set in a logical framework. However, it does not clearly expound the axiomatic theory of space actually used. The logical language itself (the ontological categories, the primitive predicates and functions) is not clearly circumscribed. Nevertheless, we believe that there are at least two reasons to clearly give such a formal framework. First, there are several alternative axiomatizations of Euclidean geometry, semantically equivalent but not ontologically nor computationally equivalent. In order to be implementable, the theory actually used must be made explicit. Second, the author uses a number of restrictions on classical topology or geometry (such as spatial regions that can be occupied by material objects must be regular) which could be integrated in the theory, yielding a less general thus more efficient representational framework. Working out an appropriate axiomatics for a theory presenting spatial properties such as those used by Davis, without taking for granted topological and geometrical theorems, is in fact the main goal of the field we have been covering up to now.

1.3.6. SPACE AND TIME OR SPACE-TIME?

Lastly, let us say a short word about approaches combining space and time. As is shown in the next section, reasoning about space often involves reasoning about change in spatial configurations, thus reasoning about space and time. When used with this purpose, the representational frameworks above have usually been extended by a separate temporal dimension or a set of temporal relations, relying on previous classical works in temporal reasoning. This is the case in Galton's chapter in this book, for instance (Chapter 10). In this quite recent work, an essential aspect of space and time integration — namely possible transitions from one spatial configuration to another or continuity of motion — is analyzed and modeled qualitatively.

However, much more is to be said to thoroughly integrate space and time in qualitative representations. First, we believe that the very fact that space and time are two clearly separated realms should be discussed. Many motion concepts — like spinning, rolling, sliding, following, going around — do not involve just change or transitions between two spatial situations. They directly hold over trajectories or "histories" in Hayes's terminology (Hayes, 1985b) — that is, four-dimensional spatiotemporal entities. Some describe the intrinsic shape of a history; others describe complex spatiotemporal relations between two histories.[17] New theories based on spatiotemporal basic entities and spatiotemporal primitive relations could be obtained. It may then be possible to define space and time — that is, purely spatial or purely temporal relations. However, it is clear that constructing such theories is really a hard problem.

Second, even if one considers space and time separately, it seems that the question of the existence of some ontological dependencies throughout the two realms should be addressed. For instance, when is it sensible to keep the following "natural" correspondence? If space is point-based, then time will be instant-based; if space is region-based, then time will be interval-based; if space is global, then time will be just another dimensional axis. It must be noted that authors do not always adopt this kind of homogeneity. On the contrary, Galton's work shows that representing continuous motion on spatial regions requires a mixed temporal ontology of instants and intervals.

To the best of our knowledge, such spatiotemporal ontological questions have only begun to be addressed, and consequently, little work has been done on qualitative representations of space-time.

1.4. Classes of Spatial Reasoning

It must be noted from the start that there is no such a thing as a "spatial logic" field, that can be compared to the temporal logic field. Problems of spatial reasoning involving the change in truth value of a proposition according to a change in location in space have been rarely tackled. von Wright (1979) constitutes a notable exception and proposes a modal logic of spatial operators such as *somewhere* or *nearby*. A very recent work sets up some requirements for modal spatial logics of this kind (Lemon, 1996). In short, spatial reasoning has little to do with a three-dimensional equivalent of reasoning on persistence, action, and change or at least, this is the case up to now.

Nevertheless, spatial reasoning can be seen as a much older field of AI than spatial representation. Much work done in path planning, in motion prediction, and in shape recognition belongs to this field without any doubt. But on the other hand, spatial representation being a far younger issue, most work in spatial reasoning has been done without using the qualitative representations of space described above. As a consequence, the correspondence and the potential mutual enrichment between the two fields have not been fully exploited. And in fact, maybe because of not having carefully considered the ontological issue of spatial representation, spatial reasoning is a field that has not been systematically structured in a widely admitted fashion. This section aims only at describing a possible way of structuring the field of spatial reasoning and not at describing in detail the work reported. Whenever possible, the focus will be given to reasoning with respect to the representational frameworks described in the previous section.

The point of view adopted here tries to group approaches with respect to how they handle spatial knowledge and spatial representations. Only in a second step does it consider classes of problems linked to application domains. In this light, the main objectives adopted by a particular approach to reasoning over spatial knowledge may be

- Exploiting the spatial information and knowledge contained in a representation (deduction),
- Changing a representation into a different format (translation and interpretation), and
- Constructing a representation according to several constraints (design).

1.4.1. DEDUCTION WITHIN REPRESENTATIONS OF SPACE

The most basic reasoning task is to exploit the spatial knowledge and spatial information encoded inside a representational framework. Two cases can be distinguished according to the completeness or incompleteness of this knowledge and information.

1.4.1.1. *Simple Deduction*

The first case of deduction aims at making explicit a fact already implicit in the representation, by exploiting properties of spatial relations, often combining two or more explicit facts. This is necessary in local spaces, where not all information is explicitly encoded, but not in global spaces in which spatial relations can be directly "read" or at least calculated numerically. The paradigmatic case of this kind of deduction, if not the most simple case, is theorem proving in axiomatic theories of space. The efficiency of such a task lies mainly on the choice of an adequate ontology. From this standpoint, higher order representations are to be avoided. But first-order theories do not necessarily guarantee tractability. In order to significantly enhance the computational ease of simple deduction, the field has generalized the use of relational algebras and *transitivity tables* or *composition tables* (see Section 1.2.1.2). Transitivity tables encode all possible compositions of relations, which have been computed once and for all. Thus, they replace theorem proving by a simple look-up operation. This method, originally developed in the field of temporal reasoning, is described and applied in Cohn et al.'s chapter of this book (Chapter 4) as well as in Freksa (1992b), Randell and Cohn (1989), Randell et al. (1992a), Egenhofer (1991), Hernández (1994), Grigni et al. (1995), Zimmermann (1993), among others.

Another way to reduce the computational complexity of general theorem proving is to apply constraint satisfaction methods, as done also in Hernández (1994) and Grigni et al. (1995). Converting first-order theories into propositional intuitionistic or modal logics is still another method that has been applied to acquire decidability (Bennett, 1994b; Nebel, 1995).

1.4.1.2. *Deduction on Partial or Uncertain Information and Extrapolation*
When information or knowledge is incomplete or uncertain, one can try to infer possible facts on the basis of hypotheses on the structure of space or space-time.

In static space, one of the main problems concerns the vagueness or the uncertainty of region boundaries or, if the ontology is boundary-free, on the vagueness of the relationship between regions. Freksa (1991) first suggested applying *neighborhood structures* previously introduced in the temporal domain (Freksa, 1992a) to the spatial domain. These structures, which indeed have many applications (see below), were assumed to serve as a basis for reasoning under uncertainty. The so-called "egg-yolk representation" considers disjunctions of possible relations between regions, building on the RCC theory and indeed exploiting neighboring properties between possible relations (Cohn and Gotts, 1996a) (see also Cohn et al.'s chapter in this book, Chapter4). Another approach applies nonmonotonic logic to spatial extrapolation, trying to "fill in" holes of uncertainty, in a fashion akin to persistence in time (Asher and Lang, 1994). This kind of work indicates a possible way to study spatial logics. Introducing the possibility of dealing with incomplete information in geographical information systems would be an important application domain.

Dealing with uncertainty in space-time has yielded a much more developed research trend — that of spatial envisionment or motion extrapolation. The aim is to simulate the spatiotemporal behavior of, in general, solid objects or more simply to predict the result of a motion. This domain of research exploits not only theories of space and time but also physical theories, such as mechanics. Three main approaches can be distinguished. The first, usually called *qualitative kinematics* adopts the general methodology of qualitative physics. State-graphs based on a particular segmentation of a Cartesian space (see Section 1.3.5) are exploited in several projects, each state-graph being highly task-dependent (Forbus, 1983; Forbus, 1995; Faltings, 1990; Forbus et al., 1991) No spatial or mechanical properties are applied in an explicit way, as the authors themselves claim to be impossible in the poverty conjecture. In addition to being thus a hardly generalizable approach, it must be noted that the complexity of simulation in the state-graphs increases rapidly with the number of static and moving objects.

The second approach applies the theorem-proving method. Davis (1988) does it in the framework of a logical model of topology, classical geometry, and mechanics (see Section 1.3.5). Nielsen (1988) exploits logical mechanical postulates in a similar framework but dramatically simplifies the coordinate space for vectors, using three values only (-,0,+), in the more classical spirit of qualitative physics. The logical approach is also adopted in naive physics, but with the aim to reason about the world in a similar fashion to how people do (Hayes, 1985b; Hayes, 1985a). The theory of mental space-time (the theory of spatial and temporal relations holding on *histories* that naive physics is supposed to use) has not been fully developed, though.

The third approach, more recent, tries to systematically capture in neighborhood structures, continuity graphs, or transition graphs, the continuity of motion as a general property of spatial relations (Hernández, 1993; Cui et al., 1992; Galton, 1993) (see also Cohn et al.'s and Galton's chapters in this book, Chapters 4 and 10). Two spatial relations are neighbors if a continuous change can yield a direct transition from one relation to the other. From these abstract graphs one can then derive the particular state-graph corresponding to a specific envisionment task. Even where orientation is taken into account, it must be noted that it is purely static orientation. Dynamic orientation is not represented, and as a consequence, reasoning on directions of motion, which is obviously a very important aspect of motion prediction, has not been tackled in this approach.

1.4.2. TRANSLATION BETWEEN SPATIAL REPRESENTATIONS

Converting information represented in one kind of spatial representation framework into another of another kind is another class of spatial reasoning.

1.4.2.1. *Language and Vision Integration.*

This area significantly overlaps the fields of computer vision, shape, and object recognition, which will not be discussed here because they are almost all based on global numerical representations of space and computational geometry algorithms (see, for example, Marr (1982), Ballard and Brown (1982), and Chen (1990). It is nonetheless interesting to mention here work aiming at integrating language and vision, an area that is growing lately (McKevitt, 1996). It draws on previous research done in both computer vision and natural language processing (understanding and generation). Clearly, the most important work on this topic has been done within the project VITRA over the last ten years (Wahlster, 1987; André et al., 1988; Herzog and Wazinski, 1994; Maass, 1995). There as in (Olivier et al., 1994), the numerical 2D representation of images is also used as the basic spatial representation for encoding the semantics of spatial expressions.

However, from a cognitive point of view as well as from a formal ontological point of view, language and vision integration raises its own classes of problems. Combining a global space, numerical or not, adequate to represent visual space, and a local space, more adequate for linguistic space, leads to interesting theoretical developments in spatial representation and, more generally, in hybrid reasoning (Habel, 1990; Latecki and Pribbenow, 1992; Habel et al., 1995; Borillo and Pensec, 1995). Work in this area, which we believe to have only begun to appear, exploits sudies on cognitive space — for instance, Herskovits's (Chapter 6) — in a systematic way. It will most probably in turn induce progress in the fields of high-level vision and the processing of spatial expressions.

1.4.2.2. *Change in Perspective or Granularity.*
It is also worth discussing here reasoning concerning change of point of view or granularity, although a single representational framework is sometimes used, and thus it could be considered as merely a special case of deduction.

Change in perspective is a frequent operation affecting orientation relations. In natural language, for instance, one often finds switches from an intrinsic reference frame to a deictic one (or the other way around), or if deictic orientation is maintained, the speaker may change in position so that the perspective point may change. These transformation are, of course, important as well in all domains where point of view is important. Hernández (1993; 1994) gives an account of how these operations can be modeled within the kind of representational framework presented in Section 1.3.1.1.

Dealing with granularity and granularity change is a very important aspect of qualitative reasoning. A few generic solutions have been proposed, and one of them applied to a (nonmetric) global space (Hobbs, 1985; Hobbs et al., 1987). In this approach, the grain is an explicit parameter. This is also the case in (Borgo et al., 1996b), where granularity is defined in a region-based local space (see Section 1.3.4.4). There, the "grain" is a particular entity chosen as a reference, identified by a predicate.

But there is another way to deal with granularity: implicitly, with operators switching between several subrepresentations. On the one hand, the domain of entities may vary in cardinality, as more or fewer spatial entities are distinguished in the different, discrete, subrepresentations. For instance, an atomic region may become nonatomic if one distinguishes several parts in it, and conversely, a nonatomic region may become atomic if its parts are no longer distinguished; the cells of the arrays may be subdivided into smaller cells or grouped into a unique bigger cell; in a nondense space of points, points may be added or removed. The hierarchical structures that have been introduced for array representations enable this kind of switch in granularity (Samet, 1984; Samet, 1989; Glasgow, 1993). A modal operator of refinement together with its converse has also been introduced on a region-based mereotopology, in particular to deal with the relativity of weak contact (Asher and Vieu, 1995) (see Section 1.3.4.1). In mereotopologies based on mixed ontologies, it would be interesting to try to model the abstraction process of seeing a region as a surface, line or point and, conversely, the refinement process of, for example, seeing a point as an extended entity. We believe that modal operators could be introduced in a similar fashion.

On the other hand, the domain of entities being fixed, the spatial relations may vary, some relations being considered as potentially coarser than others (for instance, contact taken to hold at a coarse granularity, is supposed to possibly disappear at a closer look). Euzenat (1995) exploits again neighborhood structures in an interval-based representation. Here, in fact, granularity is just another way to apprehend uncertainty.

1.4.3. DESIGN

Finally, it is impossible not to mention the class of problems related to the construction of spatial or spatiotemporal configurations filling certain requirements. This belongs to the general area of design and decision in AI and involves obtaining new spatial representations with additional entities.

This area includes one of the most important spatial problem humans have to solve a great part of their time — namely, route-finding. Indeed, for years most of the work in spatial reasoning has been done in this area. As said earlier, this large area cannot be covered here. Let us just recall that literature distinguishes between path planning at a small scale and route planning at a larger scale and add a word on the kind of spatial representations used to perform design in space-time. Path planning or motion planning uses numerical spatial representations, fitting the data robots are able to collect from their environment through sensors. Consequently, numerical global spaces are widely used. A variety of computational geometry algorithms and search algorithms operating on these representations have been developed. A classical, representative approach is based on a configuration-space representation — a dense coordinate space in which the robot's position is reduced to a point by appropriate transformations on the environment (Lozano-Pérez, 1983). Discrete global spaces (arrays) as well as local spaces based on cells and an adjacency relation, sometimes hierarchized to optimize search, have also been used (Slack and Miller, 1987; Fujimura and Samet, 1989). For a review of this area, see, for example, Latombe (1990) and Burger and Bhanu (1992). In route planning in a larger-scale environment, or navigation, the robot has to reason over spatial representations that do not need to directly match the sensor data. A classical approach is based on a *cognitive map* — a local spatial representation inspired by cognitive psychology results, whose mixed ontology combine regions, points and lines (or paths) into a network (Kuipers, 1978; Kuipers and Levitt, 1988; Levitt and Lawton, 1990). Search algorithms exploit the hierarchy created in the graph by part-whole relations over regions and incidence between points and regions. Besides route planning, gathering information to complete the cognitive map is an important issue addressed in this work as well.

Design in pure space, or space planning, is relevant to several domains of application, including architecture and land management. It covers a class of problems such as how to fit a room with furniture or how to design the plan of a new site. Here too, a variety of search methods have been applied to different kinds of spatial representations. Constraint satisfaction methods applied to local spaces have given interesting results (Baykan and Fox, 1987; du Verdier, 1993).

1.5. Conclusions

Much progress has been made in the search for new spatial ontologies and new spatial representational frameworks in the past few years, and we hope this chapter gives a good overview of this progress. Nevertheless, obtaining cognitively more adequate spatial representations is still essential for applications in natural language processing and multimodal interfaces. Constructing a complete representation of physical space in which a variety of qualitative reasoning tasks can be performed is still an open issue. What is more, reasoning on both kinds of spaces, commonsense and physical, is of a crucial importance — for instance, in geographical information systems. Looking for most generic theories of space or considering how to integrate several representations and looking for translation methods between different representations are possible ways to address this last issue. From these three points of view, new developments in spatial representation and in spatial reasoning are needed.

Several major directions of research already engaged toward this goal can be sketched:

- To look systematically for nonclassical theories of space, taking up theories already proposed in mathematics or developing new theories. This could be done with all kinds of ontologies, based on points, extended entities, and mixed ones. We believe that the full integration of topology, distance, and orientation concepts should be focused on. In particular, a first-order axiomatization of Euclidean geometry based on extended entities should be achieved. This may not guarantee that morphological concepts are easily handled in the theory, so systematic theories of shape should be developed, looking for adequate primitive concepts. With the aims of modeling mental space and obtaining a computationally effective theory, we must work out non-Euclidean geometries as well — for instance, dropping density or properly dealing with the variety of commonsense contact and boundary notions. This is one of the main lines of research adopted now in the field of qualitative spatial reasoning, of which the work described in Cohn et al.'s chapter in this book (Chapter 4) is an excellent representative. In this domain, there is an increasing awareness, if not a complete integration, of developments along a parallel line of research in formal ontology of space, for the great benefit of the discipline (see, for example, Varzi (1996a) and Casati and Varzi chapter, Chapter 3).

- To integrate and exploit non-purely spatial properties in theories of space. A way toward this goal is to consider seriously relative space — that is,theories of spatial relations on concrete entities — instead of abstract purely spatial entities. Alternatively, we could study the properties of the localizing function with respect to the properties of the entities located in an absolute space — that is to say, investigate the ontological dependencies between space and other realms such as matter. This approach may cast a new light on the choice of adequate ontologies of absolute spaces in the previous direction of research. Casati and Varzi's chapter in this book (Chapter 3) clearly opens this line of research, to which Aurnague and Vieu (1993) and Borgo et al. (1996b) have also contributed. Further investigating which ontological categories are worth considering (for instance, refining the dichotomy material-immaterial entities) may be done exploiting, for example, results in linguistics, cognitive psychology, or application domains involving physical space. Frank's and Herskovits's chapters in this book (Chapter 5 and 6) contribute toward this goal.
- To consider space-time as an integrated realm. Theories of motion based on histories as basic entities and genuine primitive spatiotemporal concepts must be developed. It is difficult to claim that this line of research has really begun, but work like that reported in Galton's chapter in this book (Chapter 10)could help start it.
- To develop spatial logics where connectives have a spatial (or spatiotemporal) meaning, not just to obtain more tractable version of first-order theories of space, but as frameworks for reasoning about space under uncertainty, or as frameworks for diagrammatic reasoning — that is, reasoning on constructing, combining, and transforming diagrams. As mentioned in Section 1.4, little work has been published in this area. We nonetheless believe it is of importance for the field in the future.

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Notes

[1] The term *relational* is also found.

[2] The domain can hold three types of entities — points, lines, and planes — with an incidence relation (not membership) between them. In Tarski (1959), an axiomatic system of Euclidean geometry based only on points is proposed; in this same system, continuity axioms are freed from any reference to arithmetic.

[3] When transposed to AI, one can see reasoning in global spaces as model-based and reasoning in local spaces as deductive. However, model-based reasoning in global spaces is in general replaced by more efficient numerical algorithmic methods.

[4] Defining "good" identity criteria is very difficult. Spatiotemporal continuity is, however, often adopted as one of these criteria, which in this case would make the definition of continuity of motion circular.

[5] For a review of possible ways of defining points in terms of regions, see Gerla (1994).

[6] When there are no boundaries (as, for instance, for the whole space), the subset is both open and closed.

[7] For instance, in the standard topology of $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, all open sets but the empty set are of the same dimensionality as the whole topological space.

[8] In the TACITUS project (Hobbs et al., 1987; Hobbs et al., 1988), a point-based global space is proposed, with independent "scales" or granular partial order relations (one for each axis).

[9] In all generality, the case $A = \pi$ should be distinguished.

[10] This means that these distance calculi require some underlying orientation system.

[11] To restrict memory size, they are sometimes compacted and hierarchically organized, as in Samet (1984; 1989).

[12] Regions extended throughout or, formally, regions x such that $\overline{x} = \overline{\overset{\circ}{x}}$ and $\overset{\circ}{x} = \overset{\circ}{\overline{x}}$.

[13] This way, a distinction can be made between jointing along a "fiat boundary" (Smith, 1995) (for example, the relation between two halves of a ball) and touching along real, objective, boundaries (for example, the relation between the ball and the ground).

[14] Note that Clarke's theory and RCC do not imply this last restriction.

[15] One may question the cognitive or physical plausibility of these particular regions. Indeed, perfect spheres may be seen as entities as abstract as points. As a consequence, a region-based geometry relying on the existence of spheres may be no more attractive than a point-based geometry.

[16] Except when there is a NTP relation between them or when they are equal, in which case orientation has no meaning. Notice that Hernández assumes that a TPP relation can be combined with orientation, considering the position of the common boundary, thus forbidding this shared boundary to be very long or to be scattered around the regions.

[17] This includes the case of the relation between one history and a static object, since immobility can be seen as being relative to a point of view.