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# Manifolds, Concepts and Moment Abstracta

## § 1 Introduction

In this paper we present a mathematical contribution to the resolution of certain metaphysical questions relating to the traditional notions of universals and abstraction. The main purpose of the paper is to introduce a theory of certain mathematical structures, called *manifolds*. We prove that one type of manifold has properties which have been historically ascribed to the (extensions of) least general universals. Within the Aristotelian and Thomistic hylemorphic ontology (a prototypical form of realist metaphysics) these universals were claimed to be hypostatisations of those forms which are the direct results of abstraction from sensible particulars, and were called *automon eide* or *infima species*. *Automon eide* were conceived within this tradition as those universals (secondary substances, or *eide*) which comprise least specific differences, and below which (in the hierarchy of generalisation) only individual (Aristotle's *primary*, or Aquinas' *composite*) substances (but not less general *eide*) fall. Considered ontologically as objects (rather than epistemologically as concepts), they may be referred to provisionally as *universal individuals of higher order*. Universal individuals of higher order are the individuals of the realm of *eide*.

This paper is the only contribution to the present volume conceived within the framework of the transcendental phenomenology inaugurated by Husserl in the *Ideas* of 1913, rather than within the realist framework of the earlier Husserl, Ingarden, *et al.* Within the transcendental tradition (which is heavily influenced by Kant's 'Copernican revolution' in metaphysics), metaphysics is identified with epistemology rather than with ontology. But in spite of the major differences in overall directions and concerns between the two traditions of Husserlian phenomenology, certain issues (including most of those confronted here) are common to both. From an *epistemological* point of view, we conceive manifolds (as defined below) as extensions of linguistically formulated concepts. We conceive concepts as intensions of unions of isomorphism types, and define a linguistically formulated concept (which we refer to as a 'predicative concept') as an intension of a manifold. By distinguishing different types of manifolds and relations between manifolds, we will therefore make possible analogous distinctions concerning predicative concepts. We shall define a linguistically formulated or predicative concept as *distinct* if and only if (iff) it is an intension of the type of manifold to be identified below as the extension of a universal individual of higher order *(automon eidos)*; any concept which is an intension of any other type of manifold we refer to as *indistinct* (or, synonymously, *vague*).

We introduce the generic term 'conceptualisation' to denote processes of attention whereby universal objects of higher order are constituted as themes, and we will formulate definitions of several types of conceptualisation discussed by Husserl and others. Definitions will be formulated in terms of types of manifolds and relations between manifolds. Because of the epistemological orientation of the paper, it is concerned with the metamathematical analysis of manifolds largely as a means to achieving such formulations concerning the constitution of perceivable objects and universals as themes of attention.

The particular context within which the paper is conceived is provided by the perspective and set of problems sketched by Aron Gurwitsch toward the end of his career. Gurwitsch was originally a student of Carl Stumpf interested in Gestalt theory and the psychology of perception. At Stumpf's suggestion he became a student of Husserl's earlier works during the 1920's. Throughout his life Gurwitsch worked as a transcendental phenomenologist who expressed a preference for the first (i.e. 'realist') edition of Husserl's *Logical Investigations,* largely because the second edition was revised to reflect Husserl's 1913 introduction of the concept of *transcendental ego* and his use of it in defining attention of various types (see Gurwitsch, 1966, chs. 10, 11). In spite of his rejection of Husserl's egological formulations, Gurwitsch remained convinced that the programme of transcendental phenomenology sketched in the *Ideas* of 1913 (but to be developed in terms of a non-egological account of attention) embodied the most promising formulation of Western metaphysics to date. He also remained convinced that the realisation of that programme demanded theoretical analysis and development of a concept introduced by Husserl in 1938, the concept of the lifeworld *(Lebenswelt)*. The following quotes are selected to sketch Gurwitsch's position, and to introduce certain key notions of his non-egological transcendental phenomenology in their native setting:

> In recent decades the theory of science has not received sufficient attention in phenomenological literature. Thus the impression could arise that phenomenology did not have much to say in that field of research, had withdrawn from it altogether to leave it to those contemporary philosophical trends, such as logical positivism, which call themselves 'scientific'. Since the necessary preparatory work was done in Husserl's later writings, the time seems to have come for phenomenology to reclaim posession of the field from which it had its departure in Husserl's earliest writings (1974, p. 31–2).

The phenomenological theory of the sciences which Gurwitsch envisioned would be built around two fundamental themes. The first of these is the life-world:

> *The first presupposition of the sciences proves to be the life-world itself, our paramount and even sole reality. Whatever unity obtains among the sciences derives from their common rootedness in the life-world,* from which all of them originate and to which most of them explicitly relate as their theme *(ibid.*, p. 139).

The second fundamental theme is consciousness or, more specifically, the processes of attention involved in the perceiving and theorising which, according to the phenomenological account, hypostatises the entities studied within the cultural, formal, and especially natural sciences:

> . . . the life-world proves to underlie and to be presupposed by the elaboration of 'objective' nature. Still, we have not yet reached the ultimate, but only the penultimate, presupposition and foundation. *The life-world,* in its turn, *refers to and, in that sense, presupposes mental life, acts of consciousness, especially perceptual consciousness through which it is experienced and presents itself as that which it is,* that is, *as that which we accept.* It is not until we arrive at consciousness as the universal medium of access (in the sense of Descertes' *Second Meditation*) to whatever exists and is valid, including the lifeworld, that our search for foundations reaches its final destination. As far as the processes of conceptualisation, idealisation, and formalisation are concerned, they now appear in their proper place as acts of consciousness of a higher order insofar as they presuppose the more elementary and more fundamental acts through which the life-world is given or, in Husserl's parlance, they are built on prepredicative experi-

ence. This is another expression of our previous conclusion that the universe of physics – objective nature as conceived by physicists – is a product of mental life constructed on the basis of the prepredicative experience of the life-world (*ibid.*, p. 58).

It is the failure to recognise the hypostatised status of the results of conceptualisation which Husserl criticised as responsible for the failure of Western metaphysics which he characterised in 1938 as the crisis of European science:

Failure to refer the accomplished products and results to the mental operations from which they derive and whose correlates they are makes one the captive of those products and results, that is, the captive of one's own creations, and that is a further aspect of traditionality. Thus, as Husserl expressed it, a cloak or tissue of ideas *(Ideenkleid)*, of mathematical ideas and symbols, is cast on the life-world to conceal it to the point of being substituted for it. What in truth is a method and the result of that method come to be taken for reality. Thus we arrive at the conception of nature . . . as possessing a mathematical structure or being a mathematical manifold (*ibid.*, p. 45).

In view of such claims on the part of (non-egological) transcendental phenomenology of science, it is not surprising that the problem of conceptualisation emerges as a central issue:

. . . *the first task of a phenomenological theory of the sciences is to develop a phenomenological theory of conceptualisation*, that is, *a phenomenological account of the transition from type to concept and eidos.*
Conceptualisation is possible along two different lines of direction which Husserl has distinguished from one another under the headings 'generalisation' and 'formalisation'. In the present context, we must confine ourselves to pointing out that the theory of conceptualisation, generalisation, formalisation, and algebraisation is one of the most urgent tasks with which phenomenological research finds itself confronted at the present stage of its development . . . the problem is far from being exhausted. Not only is there ample room for further investigation, but some of Husserl's results – we submit – require revisions and modifications (*ibid.*, p. 143).

The late Husserlian notion of the life-world is an intuitive (and effectively anthropological or sociological) notion. It was held by Gurwitsch to be fundamental to metaphysics in general and to the phenomenology of conceptualisation in particular. While the term remains at present ill-defined (see Carr, 1974, Ch. 8), it may be understood as denoting the world (and also, ambiguously, culturally relative worlds) of common sense experience i.e. the milieu(s) studied in one form or another by the various sciences of culture. Within the Husserlian tradition the work of Alfred Schutz was devoted exclusively to the task of defining (i.e. removing the vagueness of) this concept by formulating invariantly shared properties of each possible world of common sense experience. While no further mention of the lifeworld and related issues will be made in this paper, we wish to mention here that we do not conceive of the life-world as the class of *actual wholes which are perceived* (as defined in § 5 below). The definitions which we shall offer in terms of manifold theory will explicitly cover certain types of conceptualisation (e.g. *formalisation* and *generalisation*) and certain structures involved in perception (e.g. *wholes as perceived, possible* and *actual wholes which are perceived*), and they will implicitly cover the notion of *lifeworld*. We omit discussion of themes like the influence of historicity upon the material content of the lifeworld, and of spatial, temporal, and causal relations within the lifeworld. However, we consider such questions both important and approachable (elsewhere) in terms of the late Husserlian notion of *lifeworld* and the results reported here.

The theory of manifolds is of particular interest in the context of the ontology of wholes and parts as developed by Husserl, and by Mulligan, Smith, and Peter Simons elsewhere in this volume. In this paper an attempt is made to formulate a connection between manifold theory and whole-part notions by interpreting the language of wholes and parts in manifold theory. One such interpretation is specified (§ 5), and the theorems of manifold theory are translated into claims concerning whole-part relations. The purpose of this translation is to present a set of claims in order that their truth may be evaluated. These claims appear conformable to intuitions expressed in Thomistic ontology and realist phenomenology concerning wholes and parts. Moreover, one claim in particular (theorem 20) expresses a fundamental Aristotelian, Thomistic, and Scholastic tenet. The evaluation of the validity of the remaining theorems is left to those engaged in eidetic phenomenology of whole-part relations, and it is hoped that this interpretation may be suggestive for such work.

We conclude this introduction with a brief statement of the preliminary assumptions, notation and terminology we adopt in our underlying set theory, logic, and metamathematics. Further metamathematical notions, including the concepts of *second-order equivalence* and *isomorphism* are defined in § 2. §§ 3–4 contain our theory of manifolds, inspired by Husserl (1891). § 5 develops an interpretation of whole-part notions in manifold theory so that the various theorems state properties and relations of wholes and parts of various kinds. In § 6 we consider some limitations and implications of the results presented in §§ 3–4 for the problem of conceptualisation.

The underlying set theory of the metalanguage used in this discussion is a class-set theory such as those of von Neumann, Bernays, Gödel, and A. P. Morse. We adopt set-theoretic axioms and basic definitions as in Monk (1969). The primitive (undefined) terms are 'class' and 'member' or 'element'. All other concepts of our class-set theory can be defined in terms of these primitives. E.g.: A class is a *set* iff it is a member of some other class; otherwise it is a *proper class*. Braces will be used to denote finite sets whose elements are listed within the braces, and '$\varnothing$' will denote the empty set. Also, the binary operations of *intersection* ( $\cap$ ) and *union* ( $\cup$ ) of classes can be defined as usual. Finally, $A$ is a *subclass* of $B$ ($A \subseteq B$) iff each member of $A$ is a member of $B$, and $A$ is a *proper subclass* of $B$ ($A \subset B$) iff $A \subseteq B$ and not $B \subseteq A$. Manifold theory as we envision it in our ontological and epistemological interpretations cannot be captured within class-set theory (see § 5 *infra*). However, in this paper we will limit our discussion to those aspects of manifold theory which can be expressed in class-set theory, and which may be thought of as comprising the theory of *manifolds in extension*.

An *n-ary relation-in-extension* (or simply-$n$-ary relation) is a class of ordered $n$-tuples. An *equivalence relation* is a binary relation $R$ satisfying the following conditions, for all elements $x$, $y$, and $z$ to which the relation applies:

(1) reflexivity: $<x,x> \in R$
(2) symmetry: if $<x,y> \in R$, then $<y,x> \in R$
(3) transitivity: if $<x,y> \in R$ and $<y,z> \in R$, then $<x,z> \in R$

For example, the relation $R'$ such that $<x,y> \in R'$ iff $x$ and $y$ are sets with the same number of members is an equivalence relation. An equivalence relation can be thought of as a relation of similarity between objects. Equivalence relations can thus be used to classify things by means of classes of similars. For any equivalence relation $R$ and any element $x$, the *R-equivalence class* of $x$ (also called the equivalence class of $x$ under $R$) is the class of all $y$ such that $<x,y> \in R$. I.e., the $R$-equivalence class of $x$ is the class of all things similar to $x$ according to $R$.

The object languages used in this paper are those of *monadic second-order logics,* possessing all features of classical, first-order logic with equality (defined as a logical constant of the languages), but including also quantification over monadic predicate variables. We define an axiom set $\Sigma$ as a consistent set of sentences formulated in one of these monadic second-order languages. More than one object language is used because the notion of the completeness of an axiom set $\Sigma$ is defined in terms of the set of predicate constant symbols used in the sentences of $\Sigma$. The symbols which distinguish one object language from another are its predicate constants; the different object languages are thus identical, except that they differ from each other in the number and type (monadic, dyadic, etc.) of their predicate constants.

A *model* is a $(k+1)$-tuple for some ordinal $k$, consisting of a non-empty set called the *universe of the model,* followed by $k$ distinct relations on that universe. In this paper we shall further assume for each model that its universe has at least 2 elements and that for each positive integer $n$, the intersection of any finite number of its $n$-ary relations is non-empty. We assume a semantics based on Tarski (1956) for the object languages, and the notion of a sentence of an object language being true in a model iff it is satisfied by an assignment to that model (see Enderton, 1972, pp. 81–82). $M$ is a *model of an axiom set* $\Sigma$ iff each sentence of $\Sigma$ is true in $M$. E.g., let $M_3 = <\{1,2,3\}, R>$ where $R$ is the usual $\leq$ ordering on this 3-element set. Where the dyadic predicate constant '$P$' denotes $R$, and the symbol '$\approx$' denotes equality defined as a logical constant of the object languages, $M_3$ is a model of the axiom set $\Sigma_2$ consisting of the following two sentences:

1. $\forall u \exists v \neg (u \approx v)$
2. $\exists u \forall v\ Pvu$

On the other hand, let $N$ be the set of all positive integers and $R'$ be the usual $\leq$ ordering on $N$. Then $<N, R'>$ is not a model of $\Sigma_2$ because sentence 2., asserting that there exists a greatest element, is not true in $<N, R'>$. A model $M$ is *compatible* with the language $\mathfrak{L}$ iff for each positive integer $n$, $M$ has exactly as many $n$-ary relations as $\mathfrak{L}$ has $n$-adic predicate constants. We offer only the following comments concerning the syntactics of the object languages:

A *deduction system* for any object language $\mathfrak{L}$ is a system of axioms and/or inference (transformation) rules which define derivations. We define a *theorem of an axiom set* $\Sigma$ *formulated within* $\mathfrak{L}$ as any sentence of $\mathfrak{L}$ which results from a derivation from $\Sigma$. Two properties defined on deduction systems are:

1. *Correctness:* Any deduction system is *correct* iff for all object language sentences $\varphi$, if $\varphi$ is a theorem of $\Sigma$ then $\varphi$ is true in each model of $\Sigma$.
2. *Completeness:* Any deduction system is *complete* iff for all object language sentences $\varphi$, if $\varphi$ is true in each model of $\Sigma$, then $\varphi$ is a theorem of $\Sigma$.

We assume any *correct* deduction system for monadic second-order logic with identity that extends a *correct* and *complete* deduction system for first-order logic with identity for our object languages. Further, for any axiom set $\Sigma$ consisting of sentences formulated in $\mathfrak{L}$, $\Sigma$ is *complete* iff for each sentence $\varphi$ of $\mathfrak{L}$, either $\varphi$ or $\neg\ \varphi$ is a theorem of $\Sigma$.

These assumptions are common conventions in contemporary logic. While space will not permit their further elaboration here, they may be found more fully described in many currently available textbooks (e.g. Enderton, 1972).

## § 2 Isomorphism Types

This section consists mainly of a brief list of definitions of the basic metamathematical concepts needed for the present work. The section concludes with three theorems establishing that isomorphism is an equivalence relation, and that isomorphism types and second-order equivalence types are proper classes. These facts (along with theorem 4) are included here in order to establish the relationship of the universe of discourse of manifold theory to that of class-set theory; they are neither proven nor used in this work.

Two models $M$ and $M'$ which are compatible with the same languages are said to be *isomorphic* iff there is a one-to-one function $f$ whose domain is the universe of $M$, whose range is the universe of $M'$, and which preserves the relations of $M$ in the following sense: the function *f preserves the relations of M* (maps $M$ isomorphically to $M'$) iff for each $n$-tuple $< a_1, \ldots, a_n >$ of elements of $M$ and for each $n$-ary relation

$R$ of $M$, $< a_1, \ldots a_n > \in R$ iff $< f(a_1), \ldots, f(a_n) > \in R'$, where $R'$ is the relation in $M'$ corresponding to $R$. Roughly speaking, $M$ and $M'$ are isomorphic just in case there is a 1-1 correspondence between their universes and between their relations such that corresponding elements are in the corresponding relations.

The relation(-in-extension) of isomorphism is the class of all ordered pairs of isomorphic models. I.e. $< M, M' > \in$ isomorphism iff $M$ and $M'$ are isomorphic. An axiom set is *categorical* iff any two models of it are isomorphic.

**Theorem 1:** Isomorphism is an equivalence relation.

Mathematicians refer to equivalence classes under the isomorphism relation as 'isomorphism types'. An *isomorphism type* is the class of all models isomorphic to a particular model. Isomorphism types are philosophically interesting for a number of reasons. Every manifold is a union of isomorphism types (theorem 13, *infra*), and ordinals can be defined as isomorphism types of a certain sort ( *viz.* well-ordering types; see Wilder, 1952, ch. 5). Since Husserl and Gurwitsch identified formalisation as the type of conceptualisation which is involved in the hypostatisation of the finite and first transfinite ordinals, we have elected to define this type of conceptualisation in terms of isomorphism types. In view of the use we intend to make of isomorphism types, their class-theoretic status is worth stating as a theorem:

**Theorem 2:** Each isomorphism type is a proper class.

Theorem 2 can be proven using the set-theoretic axioms of substitution (also called 'replacement') and regularity (Monk, 1969, p. 180).

Two models $M$ and $M'$ are *monadic second-order equivalent* (in symbols $M \equiv_{m2} M'$ ) iff exactly the same sentences are true in each of them; i.e. $M \equiv_{m2} M'$ iff each sentence true in either one of these two models is also true in the other. $M$ and $M'$ are *elementarily equivalent* (in symbols $M \equiv M'$) iff exactly the same *elementary sentences* (sentences containing no predicate variables) are true in each of these models. The $\equiv_{m2}$-type of a model $M$ is the class of models which are monadic second-order equivalent to $M$, and the $\equiv$-type of a model $M$ is the class of models which are elementarily equivalent to $M$.

**Theorem 3:** Each monadic second-order equivalence type ($\equiv_{m2}$-type) is a proper class.

Theorem 3 can be proven from theorem 2 by observing that each $\equiv_{m2}$-type contains an isomorphism type as a subclass. We include the statements of theorems 2 and 3 in order to make it clear that any theory of isomorphism types, $\equiv_{m2}$-types, and manifolds must make statements attributing properties to and imposing relations on proper classes. We now develop some fundamental notions of one such theory.

## § 3 Manifolds

This section deals primarily with one (extensional) version of the theory of manifolds, a theory similar to (and inspired by) the theory of manifolds developed by Edmund Husserl in *Philosophie der Arithmetik* and subsequent works. The following quotes, taken from the (amended) English translation of *Formale und transzendentale Logik*, sketch the notions of *manifold* and *definite manifold* developed first in *Philosophie der Arithmetik*.

The first passage introduces Husserl's notion, and shows clearly that he had a metamathematical context in mind when speaking of manifolds; the relationship between what he called 'formal apophantics' (the study of 'judgment-forms, . . . proof forms, . . . *judgment-systems in their entirety'*) and *('auf der gegenständlichen Seite')* 'formal ontology' (the study of 'any objects whatever, any set and any set-relationship whatever; any combinations, orders, magnitudes, or the like, . . . of objectual totalities (manifolds)') is analogous (to say the least) to the metamathematical relationship between logic and axiomatics on the one hand, and model theory on the other.

. . . a beginning was found here for a theory of deductive systems or, in other words, a logical discipline relating to the deductive sciences as such and considered as theoretical *wholes*. As the earlier level of logic had taken for its theme the pure forms of all meaning formulations that, as a matter of *a priori* possibility, can occur within a science: judgment-forms (and the forms of their elements), argument-forms, proof-forms – correlatively (on the objectual side): any objects whatever, any set and any set-relationship whatever; any combinations, orders, magnitudes, or the like, with the pertinent formal essential relationships and connections. So now *judgment-systems in their entirety* become the theme – systems each of which makes up the unity of a possible deductive theory, a (possible) 'theory in the strict sense'. As the concept of an objectual totality (a concept always understood in formal generality), there appears here that which mathematics, without any explicative determination of its sense, has in mind under the name *'Mannigfaltigkeit* (manifold)'. It is the form-concept of the object-realm of a deductive science, this being thought of as a systematic or total unity of theory (§ 28).

Husserl's references to the originator of differential geometry as the source of the concept of *manifold* leave no doubt concerning the metamathematical nature of the notion:

The great advance of modern mathematics, particularly as developed by Riemann and his successors, consists not merely in its having made clear to itself the possibility of going back in this manner to the form of a deductive system . . . but rather in its having also gone on to view *such system-forms themselves as mathematical objects* . . . (§ 30).

Where Husserl appears to have been working with an inadequate distinction between the manifold and a model of an axiom set, our definitions will respect this distinction, and may therefore be viewed as a slight but perhaps useful variation on Husserl's work. We are nevertheless able to establish enough of the properties attributed by Husserl to manifolds that we believe the definitions proposed here clarify his own ideas. To see this relationship between our results and those of Husserl, it is useful to consider some of his comments regarding the type of manifold which he characterised as *definite:*

The tendency towards a preeminent version of the mathematical concept of the manifold (and therefore toward one particular aim in the theory of manifolds) was determined by the Euclidean ideal. I attempted to give that version concrete formulation in the *concept of the definite manifold.*

The hidden origin of this concept, which, it seems to me, has continually guided mathematics from within, is as follows. If we imagine the *Euclidean ideal* as realised, then the whole infinite system of space-geometry would be derivable from an irreducible finite system of axioms by purely syllogistic deduction (that is to say, according to the principles of the lower level of logic); and thus the *a priori* essence of space would be *capable of becoming completely disclosed in a theory.* The transition to form thus yields the form-idea of any manifold whatever that, conceived as subject to an axiom-system with the *form* derived from the Euclidean axiom-system by formalisation, could be *completely explained nomologically,* and indeed in a deductive theory that would be (as I used to express it in my Göttingen lectures) 'equiform' with geometry. If a manifold of indeterminate generality is conceived from the start as defined by such a system of forms of axioms – if it is conceived as determined exclusively thereby – then the wholly determinate system of the forms belonging to the theorems and component theories, and ultimately the whole *science-form* necessarily valid for such a manifold, can be derived by pure deduction. Naturally all the concretely exhibited material manifolds subject to axiom-systems that, on being formalised, turn out to be equiform, have the same deductive science-form in common; they are equiform precisely in relation to this deductive science-form (§ 31).

It may be noted that the term 'equiform' used by Husserl is the Latin translation of the Greek 'isomorphic', and that the notions of isomorphism and formalisation play fundamental roles in Husserl's conception of the definite manifold. However, in view of the references made by Husserl to Hilbert in these contexts, and of Husserl's suggestions concerning the equivalence of the property of completeness of an axiom set and the definiteness of its manifold, we have elected to use a more general notion than isomorphism in defining the property of Husserl-definiteness:

In proceeding from such considerations of the peculiar nature of a nomological object-realm to formalisation, there was yielded that which is pre-eminently distinctive of a ***manifold-form in the pregnant sense,*** i. e. in the sense of a form that is nomologically explicative. Such a manifold-form is defined ***not by just any formal axiom-system*** but by a *'complete'* one. Reduced to the precise form of the concept of the definite manifold, this implies:
That the axiom-system formally defining such a manifold is distinguished by this, that every proposition (proposition-form, of course) that can be constructed, in accordance with the grammar of pure logic, out of the concepts (concept-forms) occurring in that system, is either 'true', i.e. is an analytic (purely deductive) consequence of the axioms, or 'false', i.e. is an analytic contradiction: *tertium non datur* . . .
Throughout the present exposition I have used the expression 'complete system of axioms', which was not mine originally but derives from Hilbert . . . The analyses given above should make it clear that the inmost motives that guided him mathematically were, even though inexplicitly, tending essentially in the same direction as those that determined the concept of the definite manifold (§ 31).

In the development of the definitions of the present section, we have chosen to take as fundamental Husserl's claim concerning the definiteness of the manifold of any complete axiom set, and will therefore define definite manifolds as $\equiv_{m2}$-types, rather than as isomorphism types. However, it will be found that some Husserl-definite manifolds (including the type which are most philosophically significant; see theorem 22, *infra*) are isomorphism types.

In this section we develop definitions of *manifold*, Husserl-*definite manifold*, and *formal manifold*, as well as theorems stating several properties of these manifolds. Theorem 4 states that each manifold is a proper class. Theorems 5 through 10 deal with the fundamental properties which Husserl suggested hold for definite manifolds, most notably that a manifold of an axiom set is definite iff the axiom set is complete. The "if" direction of this property is established in theorem 6. Theorem 8, however, provides a counter-example showing that the "only if" direction does not hold. Theorem 9 is a weaker version of the "only if" condition which does hold, and theorem 10 establishes the full strength of Husserl's property for first-order logic.

The *manifold of an axiom set* $\Sigma$ (denoted by '$\mathfrak{M}\Sigma$') is the class of all models of $\Sigma$ which are compatible with the language whose only predicate constants are equality and those appearing in at least one sentence of $\Sigma$.

**Theorem 4:** Each manifold is a proper class.

Theorem 4 can be proved from theorem 2 by observing that each $\equiv_{m2}$-type contains an isomorphism type as a subclass.

For any manifold $\mathfrak{M}$, $\mathfrak{M}$ is *Husserl-definite* iff $\mathfrak{M}$ is an $\equiv_{m2}$-type. In the following theorems we prove several of the fundamental properties claimed by Husserl (1891) for Husserl definite manifolds.

**Theorems 5:** Each manifold which is an isomorphism type is Husserl-definite.

**Proof:** Let $\mathfrak{M}\Sigma$ be any manifold which is an isomorphism type. We shall show that $\mathfrak{M}\Sigma$ is an $\equiv_{m2}$-type by showing that (1) for each model $M$ in $\mathfrak{M}\Sigma$, and each model $M'$, if $M' \equiv_{m2} M$, then $M' \varepsilon \mathfrak{M}\Sigma$; and (2) if $M \varepsilon \mathfrak{M}\Sigma$ and $M' \varepsilon \mathfrak{M}\Sigma$, then $M \equiv_{m2} M'$. The hypotheses of (1) imply that every sentence of $\Sigma$ is true in $M$ and that $M'$ has the same true sentences as $M$. Thus $M'$ is a model of $\Sigma$ and the conclusion of (1) follows. The hypotheses of (2) along with the assumption that $\mathfrak{M}\Sigma$ is an isomorphism type imply that $M$ and $M'$ are isomorphic. Thus $M \equiv_{m2} M'$. Consequently $\mathfrak{M}\Sigma$ is an $\equiv_{m2}$type, and hence a Husserl-definite manifold.

We define any manifold which is an isomorphism type as a *formal manifold*. Thus, some Husserl-definite manifolds are formal manifolds, and every formal manifold is Husserl-definite by theorem 5.

**Theorem 6:** For each complete axiom set $\Sigma$, the manifold $\mathfrak{M}\Sigma$ is Husserl-definite.

**Proof:** Let $\Sigma$ be any complete axiom set. Again, we show that $\mathfrak{M}\Sigma$ is an $\equiv_{m2}$-type by proving that (1) for each model $M \in \mathfrak{M}\Sigma$, if $M' \equiv_{m2} M$ then $M' \in \mathfrak{M}\Sigma$; and (2) if $M \in \mathfrak{M}\Sigma$ and $M' \in \mathfrak{M}\Sigma$, then $M \equiv_{m2} M'$. As in the preceding proof, (1) is clear since its hypotheses imply that every sentence of $\Sigma$ is true in $M$ and that $M'$ has the same true sentences as $M$. To prove (2), observe that the hypotheses imply that each sentence in $\Sigma$ is true in $M$ as well as in $M'$. Since $\Sigma$ is complete, for each sentence $\varphi$ of the language of $\Sigma$, either $\varphi$ or $\neg\varphi$ is a theorem of $\Sigma$. Thus by the correctness property, either $\varphi$ or $\neg\varphi$ is true in every model of $\Sigma$. Hence $M$ and $M'$ have the same true sentences, and thus $M \equiv_{m2} M'$.

**Lemma 7:** For any axiom set $\Sigma$, if $\Sigma$ is categorical, then $\mathfrak{M}\Sigma$ is a Husserl-definite, formal manifold.

**Proof:** Let $\Sigma$ be any categorical axiom set. Since $\Sigma$ is categorical, $\mathfrak{M}\Sigma$ is a manifold and an isomorphism type. The result now follows by theorem 5.

Husserl claimed the converse of theorem 6 (see preceding quote), and seemed to suggest theorem 6 as properties of definite manifolds. But using Gödel's incompleteness theorem of 1931, we show that the converse of theorem 6 does *not* hold:

**Theorem 8:** There exists a (categorical) set $\Sigma$ of monadic second-order sentences such that the manifold $\mathfrak{M}\Sigma$ is Husserl-definite, but $\Sigma$ is not complete.

**Proof:** Let $\Sigma_1$ be the set of the following seven axioms for the non-negative integers with '$<$' and '*S*' respectively denoting the *less than relation* and the *successor operation*.

**Axiom 1:** $\forall u \forall v\{\neg (u \approx v) \rightarrow [(u < v) \vee (v < u)]\}$

**Axiom 2:** $\forall u \forall v[(u < v) \rightarrow \neg (u \approx v)]$

**Axiom 3:** $\forall u \forall v \forall w\{[(u < v) \wedge (v < w)] \rightarrow (u < w)\}$

**Axiom 4:** $\forall V\{\exists u Vu \rightarrow \exists v\langle Vv \wedge \forall w\{Vw \rightarrow [(v < w) \vee (v \approx w)]\}\rangle\}$

**Axiom 5:** $\neg \exists u \forall v[(v < u) \vee (v \approx u)]$

**Axiom 6:** $\forall u\langle \exists v(v < u) \rightarrow \exists u_1 \{(u_1 < u) \wedge \neg \exists u_2[(u_1 < u_2) \wedge (u_2 < u)]\}\rangle$

**Axiom 7:** $\forall u \forall v\langle (Su \approx v) \leftrightarrow \{(u < v) \wedge \neg \exists w [(u < w) \wedge (w < v)]\}\rangle$

Let $\Sigma_2$ be a set of axioms for addition and multiplication of natural numbers, such as axioms M1, M2, E1, and E2 in Enderton (1972, p. 194). And let $\Sigma = \Sigma_1 \cup \Sigma_2$. $\Sigma$ is categorical, and hence $\mathfrak{M}\Sigma$ is Husserl-definite. But $\Sigma$ is not complete, by Gödel's incompleteness theorem (1931). In effect, since $\mathfrak{M}\Sigma$ is Husserl-definite, there is a complete set $\Sigma'$ of sentences such that $\mathfrak{M}\Sigma = \mathfrak{M}\Sigma'$. (E.g. let $\Sigma'$ be the set of all sentences true in some particular model of $\Sigma$). Moreover, $\Sigma \subseteq \Sigma'$, but their *equality* would be equivalent to a completeness theorem, which does not hold in monadic second-order logic.

Nevertheless, we are able to provide the following weaker relationship between the definiteness of a manifold and the completeness of some axiom set for it:

**Theorem 9:** For all manifolds $\mathfrak{M}$, if $\mathfrak{M}$ is Husserl-definite, then there is some perhaps infinite axiom set $\Sigma$ such that $\mathfrak{M}$ is the manifold of $\Sigma$ and $\Sigma$ is complete.

**Proof:** Assume $\mathfrak{M}$ is a Husserl-definite manifold. For any model $M \in \mathfrak{M}$, let $\Sigma$ be the set of all sentences which are true in $M$. Then for each sentence $\varphi$ of the language compatible with $M$, either $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$. Thus $\Sigma$ is complete. To show that $\mathfrak{M} = \mathfrak{M}\Sigma$, observe that $\mathfrak{M}$ is the set of all models which are monadic second-order equivalent to $M$, and hence $\mathfrak{M}$ is the set of all models which have the same true sentences as $M$.

It should be pointed out that Husserl's comments indicate that he never considered the notion of an infinite axiom set, and would probably have found it philosophically unsatisfying. This development of manifold theory is based on monadic second-order object languages. Such languages are used here for two reasons: (1) Husserl seemed to be conceiving of 2nd-order entities in his concern with some objects of higher order, and (2) we intend to apply these results to a (noetic) foundation of mathematics which will be based on second-order axioms of well-orderings (see now Null and Simons, 1981). The proof of theorem 8 depends on the use of a second-order sentence (axiom 4). Moreover, if we restrict to first-order languages, it can be proven that an axiom set is complete iff its manifold is definite.

A *first-order Husserl-definite manifold* is a manifold which is an equivalence type under the relation of elementary equivalence ( $\equiv$ ). Also, a *first-order axiom set* is an axiom set containing no sentence in which a predicate variable occurs, and a *first-order complete axiom set* is a first-order axiom set $\Sigma$ such that for each first-order (elementary) sentence $\varphi$ of the language of $\Sigma$, either $\varphi$ or $\neg\varphi$ is a theorem of $\Sigma$.

**Theorem 10:** For each first-order axiom set $\Sigma$, $\mathfrak{M}\Sigma$ is a first-order, Husserl-definite manifold iff $\Sigma$ is first-order complete.

**Proof:** Let $\Sigma$ be any first-order axiom set. First, assume that $\mathfrak{M}\Sigma$ is a first-order, Husserl-definite manifold containing the model $M$. I.e. all models of $\Sigma$ are elementarily equivalent to $M$. Let $\varphi$ be any first-order sentence formulated in the language of $\Sigma$, and clearly either $\varphi$ is true in $M$ or $\neg\varphi$ is true in $M$. Hence either $\varphi$ is true in each model of $\Sigma$ or $\neg\varphi$ is true in each model of $\Sigma$. Now, by the completeness theorem of first-order logic, either $\varphi$ or $\neg\varphi$ is a theorem of $\Sigma$. Thus $\Sigma$ is first-order complete. Secondly, we must show that if $\Sigma$ is first-order complete, then $\mathfrak{M}\Sigma$ is a first-order, Husserl-definite manifold. This can be proven by the argument used for theorem 6, with '$\equiv_{m2}$' replaced by '$\equiv$' and terms like 'sentence' and 'complete' referring to their first-order cases.

We claim that theorems 5 through 10 show that we have formulated a close approximation to the notion of *definite manifold* which Husserl had in mind, and that the disparity between our description and his shown by theorems 8 through 10 is a result of the correction of his assumption of completeness of second-order logic. Some further disparities may be expected as a result of our distinction between *models* and *manifolds* of axiom sets. A further property (which we shall refer to as the *automon eidos* property) claimed by Husserl for definite manifolds will be proven (along with certain other properties) in the next section.

## § 4 Further Aspects of Manifold Theory

This section begins with a lemma establishing a technical fact about equivalence classes which is to be used only in proving theorems 12 and 13. Theorems 12 through 24 state important relationships among manifolds, Husserl-definite manifolds, isomorphism types, and certain types of models. They climax with theorems 18–24, which indicate that one type of Husserl-definite manifold is a mathematical structure which exhibits properties historically ascribed (within the Aristotelian and Thomistic traditions) to least general universals (i.e. to *automon eide*).

**Lemma 11:** Let $\mathfrak{M}$ be any class and $E$ be any equivalence relation defined on $\mathfrak{M}$. For each $M \in \mathfrak{M}$, let $E_M$ be the equivalence class under $E$ which contains $M$ as a member. If for each $M \in \mathfrak{M}$, $E_M$ is a subclass of $\mathfrak{M}$, then $\mathfrak{M}$ equals an $E$-equivalence class or a union of at least two disjoint $E$-equivalence classes.

**Proof:** Let $U$ be the union of all $E_M$ such that $M \in \mathfrak{M}$. Clearly, each $M \in \mathfrak{M}$ is a member of its $E_M$, and thus $\mathfrak{M} \subseteq U$. By hypothesis each $E_M \subseteq \mathfrak{M}$, and thus the union $U$ of all those $E_M$'s is a subclass of $\mathfrak{M}$. Hence $U = \mathfrak{M}$. If all those $E$-equivalence classes are equal, $\mathfrak{M} =$ that $E$-equivalence class. Otherwise $U$ (and hence $\mathfrak{M}$) can be written as a union of *disjoint* equivalence classes (because two equivalence classes are either equal or disjoint).

The next two theorems are proven by applying Lemma 11, using $\equiv_{m2}$-equivalence and isomorphism, respectively, as the equivalence relation $E$.

**Theorem 12:** Each manifold is either Husserl-definite or is a union of at least two disjoint Husserl-definite manifolds.

**Proof:** Given any manifold $\mathfrak{M}\Sigma$ and any $M \in \mathfrak{M}\Sigma$, let $\mathfrak{M}_M$ be the Husserl-definite manifold containing $M$, and we shall show that $\mathfrak{M}_M \subseteq \mathfrak{M}\Sigma$. Let $M'$ be any member of $\mathfrak{M}_M$. Then $M \equiv_{m2} M'$, and hence $M$ and $M'$ have the same true sentences. Thus, since $M$ is a model of $\Sigma$, $M'$ is also. I.e. $M' \in \mathfrak{M}\Sigma$, establishing that each $\mathfrak{M}_M$ is a subclass of $\mathfrak{M}\Sigma$. The conclusion now follows by lemma 11.

**Theorem 13:** Each manifold is either an isomorphism type or a union of (disjoint) isomorphism types.

**Proof:** Let $M$ be any member of the manifold $\mathfrak{M}\Sigma$. Let $I_M$ be the isomorphism type containing $M$ as an element. We shall show that $I_M \subseteq \mathfrak{M}\Sigma$. Let $M'$ be any member of $I_M$. Then $M$ and $M'$ are isomorphic, and thus they have the same true sentences. Hence, since $M$ is a model of $\Sigma$, $M'$ is also, and $M' \in \mathfrak{M}\Sigma$. Thus $I_M \subseteq \mathfrak{M}\Sigma$ and the conclusion follows by lemma 11.

**Theorem 14:** (a) There exists an isomorphism type which is not a formal manifold; (b) there exists a manifold which is neither a formal manifold nor a union of formal manifolds; (c) there exists a Husserl-definite manifold which is neither a formal manifold nor a union of formal manifolds.

**Proof:** Let $\Sigma_{\text{w.o.}}$ consist of axioms 1–4 in the proof of theorem 8. Then $\mathfrak{M}\Sigma_{\text{w.o.}}$ (the manifold of well-ordered models) has an isomorphism type for each ordinal number, and each of those isomorphism types is a subset of $\mathfrak{M}\Sigma_{\text{w.o.}}$. In particular, $\mathfrak{M}\Sigma_{\text{w.o.}}$ contains more than $2^{\aleph_0}$ isomorphism types. However, there are only $\aleph_0$ many sentences in the language of $\Sigma_{\text{w.o.}}$ and thus at most $2^{\aleph_0}$ many sets of sentences. Thus at most $2^{\aleph_0}$ many of the isomorphism types contained in $\mathfrak{M}\Sigma_{\text{w.o.}}$ can be manifolds of the form $\mathfrak{M}\Sigma$ for some axiom set $\Sigma$ in the same language as $\Sigma_{\text{w.o.}}$. I.e. at most $2^{\aleph_0}$ many isomorphism types of $\mathfrak{M}\Sigma_{\text{w.o.}}$ are formal manifolds. Thus the manifold $\mathfrak{M}\Sigma_{\text{w.o.}}$ contains more isomorphism types than formal manifolds, which means that some isomorphism type $I$ of $\mathfrak{M}\Sigma_{\text{w.o.}}$ is not a formal manifold, establishing (a). By theorem 13, $\mathfrak{M}\Sigma_{\text{w.o.}}$ is the union of its isomorphism types. Since distinct isomorphism types are disjoint, the members of $I$ are absent from the union of the formal manifolds of $\mathfrak{M}\Sigma_{\text{w.o.}}$. Thus $\mathfrak{M}\Sigma_{\text{w.o.}}$ is not a union of formal manifolds of $\mathfrak{M}\Sigma_{\text{w.o.}}$. Thus $\mathfrak{M}\Sigma_{\text{w.o.}}$ is not a union of formal manifolds, completing the proof of (b). To prove (c), let $M$ be any model in $I$ and let $\mathfrak{M}$ be the Husserl-definite manifold containing $M$. By a proof similar to that of theorem 13, we can show that $I \subseteq \mathfrak{M}$. Since $I$ is not a manifold, $\mathfrak{M} \neq I$. Thus $I \subset \mathfrak{M}$ and by theorem 13, $\mathfrak{M}$ is a union of its isomorphism types. Again, these isomorphism types are disjoint, and hence the members of $I$ are absent from the union of the formal manifolds of $\mathfrak{M}$, establishing (c).

**Theorem 15:** Each isomorphism type is a subclass of some Husserl-definite manifold, but is not necessarily itself a manifold.

**Proof:** Let $I$ be any isomorphism type, let $M$ be any model in $I$, and let $\mathfrak{M}$ be the Husserl-definite manifold containing $M$. We shall show that $I \subseteq \mathfrak{M}$. Let $M'$ be any member of $I$. Then $M$ and $M'$ are isomorphic, and hence have the same true sentences. I.e. $M \equiv_{m2} M'$, and thus $M' \in \mathfrak{M}$, establishing that $I \subseteq \mathfrak{M}$. Moreover, by theorem 14 (a), $I$ is not necessarily a manifold.

An *expansion* of a model $M$ is a model $M'$ having the same universe as $M$ such that the set of relations on $M$ is a proper subset of the set of relations on $M'$, and a *reduct* of a model $M'$ is a model $M$ such that $M'$ is an expansion of $M$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be any manifolds. First, $\mathfrak{M}_1$ is *a generalisation of* $\mathfrak{M}_2$ *by weakening axioms* iff $\mathfrak{M}_2 \subset \mathfrak{M}_1$. In such a case, $\Sigma_1$ is *weaker than* $\Sigma_2$ in the sense that each sentence of $\Sigma_1$ is a consequence of $\Sigma_2$, while some sentence of $\Sigma_2$ is not a consequence of $\Sigma_1$. Secondly, $\mathfrak{M}_1$ is *a generalisation of* $\mathfrak{M}_2$ *by removals* iff there exist two sets of sentences $\Sigma_1$ and $\Sigma_2$ such that $\mathfrak{M}_1 = \mathfrak{M}\Sigma_1$, $\mathfrak{M}_2 = \mathfrak{M}\Sigma_2$, $\Sigma_1 \subset \Sigma_2$, and each model in $\mathfrak{M}_2$ is an expansion of some model in $\mathfrak{M}_1$. This style of generalising $\mathfrak{M}_2$ involves removing axioms from an axiom set $\Sigma_2$ for $\mathfrak{M}_2$ and removing relations from each model in $\mathfrak{M}_2$ along with corresponding predicates from the language of $\Sigma_2$. Last, we define $\mathfrak{M}_1$ as *a generalisation of* $\mathfrak{M}_2$ iff it is one or the other kind of generalisation defined above, or there exists a manifold $\mathfrak{M}_3$ such that $\mathfrak{M}_3$ is a generalisation of $\mathfrak{M}_2$ by removals, and $\mathfrak{M}_1$ is a generalisation of $\mathfrak{M}_2$ by weakening axioms.

**Theorem 16:** For each isomorphism type $I$ there exists a Husserl-definite manifold $\mathfrak{M}$ such that $I$ is a subclass of $\mathfrak{M}$ and of every generalisation of $\mathfrak{M}$ by weakening axioms.

**Proof:** Consider any isomorphism type $I$. By theorem 15, $I$ is a subclass of some Husserl-definite manifold $\mathfrak{M}$. For each generalisation by weakening axioms $\mathfrak{M}'$ of $\mathfrak{M}$, we have $\mathfrak{M} \subset \mathfrak{M}'$, and hence $I \subseteq \mathfrak{M}'$. Thus $I$ is a subclass of $\mathfrak{M}$ and of each generalisation of $\mathfrak{M}$ by weakening axioms.

**Theorem 17:** Each Husserl-definite manifold can be generalised by weakening axioms to a manifold (which by the next theorem is not Husserl-definite).

**Proof:** Let $\mathfrak{M}\Sigma$ be any Husserl-definite manifold. By vacuous quantification, for each model $M$, every axiom in the null axiom set is true in $M$. Thus each model is a model of the null axiom set $\Sigma_\varnothing$. Hence $\mathfrak{M}\Sigma \subseteq \mathfrak{M}\Sigma_\varnothing$. Moreover, for each language, there exist models $M$ and $M'$ compatible with that language such that $M \not\equiv_{m2} M'$. Thus $\mathfrak{M}\Sigma_\varnothing$ is not an $\equiv_{m2}$-type, and hence $\mathfrak{M}\Sigma \subset \mathfrak{M}\Sigma_\varnothing$.

In view of the proof tactic for theorem 17, it is worth pointing out that there are also manifolds $\mathfrak{M}\Sigma'$ such that $\mathfrak{M}\Sigma'$ is a generalisation by weakening axioms of $\mathfrak{M}\Sigma$ and $\mathfrak{M}\Sigma_{\varnothing}$ is a generalisation by weakening axioms of $\mathfrak{M}\Sigma'$.

**Theorem 18:** A manifold is Husserl-definite iff there exists no manifold of which it is a generalisation by weakening axioms.

**Proof:** First, assume that $\mathfrak{M}$ is a Husserl-definite manifold, and for a *reductio ad absurdum* proof, assume that $\mathfrak{M}$ is a generalisation by weakening axioms of the manifold $\mathfrak{M}\Sigma$. Thus $\mathfrak{M}\Sigma \subset \mathfrak{M}$. Since each model $M$ in $\mathfrak{M}\Sigma$ is a member of $\mathfrak{M}$ and $\mathfrak{M}$ is a $\equiv_{\mathrm{m2}}$-type, $M$ is $\equiv_{\mathrm{m2}}$ to every model in $\mathfrak{M}$. Thus each model in $\mathfrak{M}$ has the same true sentences as $M$, which is a model of $\Sigma$. Hence each model in $\mathfrak{M}$ is a model in $\mathfrak{M}\Sigma$. I.e. $\mathfrak{M} \subseteq \mathfrak{M}\Sigma$, contradicting the reductio assumption that $\mathfrak{M}\Sigma \subset \mathfrak{M}$. Secondly, assume $\mathfrak{M}$ is a manifold which is not the generalisation by weakening axioms of any manifold. For a *reductio ad absurdum* proof, assume $\mathfrak{M}$ is not a Husserl-definite manifold. Then by theorem 12, $\mathfrak{M}$ is a union of at least two Husserl-definite manifolds. If $\mathfrak{M}'$ is one of them then $\mathfrak{M}' \subset \mathfrak{M}$, and thus $\mathfrak{M}$ is a generalisation of $\mathfrak{M}'$ by weakening axioms, contradicting the assumption.

Define a *fully expanded model* as a model $M$ such that for each positive integer $n$, the set $S_n$ of all $n$-ary relations in $M$ is an *ultrafilter.* Such an $S_n$ is an *ultrafilter* iff:

(i) $S_n \neq \varnothing$

(ii) $\varnothing \notin S_n$

(iii) Whenever $R \in S_n$ and $R' \in S_n$; $R \cap R' \in S_n$

(iv) Whenever $R \in S_n$, $R \subseteq R'$ and $R'$ is an $n$-ary relation on $U$ then $R' \in S_n$

(v) Whenever $R$ is an $n$-ary relation on $U$ then either $R$ or its complement is a member of $S_n$ (where the complement of $R$ = $\{(a_1,\ldots,a_n) \mid \text{each } a_i \in U \text{ and } (a_1,\ldots,a_n \notin R\}$.

**Lemma 19:** (a) No fully expanded model has an expansion. (b) Every model is fully expanded or has an expansion which is.

**Proof:** (a) Let $M$ be any fully expanded model. Suppose we attempt to expand $M$ to $M'$ by including the additional $n$-ary relation $R$. If $R$ is not a relation of $M$, then the complement of $R$ is a relation of $M$. Thus $M'$ has two relations, $R$ and its complement, whose intersection is empty. Hence, $M'$ is not a model by our definition. If $R$ is a relation of $M$, then the relation $R$ appears twice in $M'$. Such an $M'$ does not fit our definition of a model, because not all of its relations are distinct.

(b) Let $M$ be any model which is not fully expanded. Let $M_2$ be a model defined as follows: (1) $M$ and $M_2$ have the same universe; (2) for each $n$, $M_2$ contains the intersection of any finite number of $n$-ary relations of $M$, and (3) $M_2$ contains each relation which is defined on its universe and which includes at least one of the above intersections as a subset. Then $M_2$ is $M$ or an expansion of $M$. Moreover, for each $n$, the set of all $n$-ary relations of $M_2$ is either empty or satisfies properties (i) – (iv) of the definition of an ultrafilter and hence is a 'filter'. By the ultrafilter theorem (Kopperman, 1972, p. 76) each filter is a subset of some ultrafilter. Now let $M_3$ be a model with the same universe as $M_2$ obtained as follows: For each $m$, select an ultrafilter $F_n$ containing all $n$-ary relations of $M_2$, and let $M_3$ contain all relations in $F_n$. Then $M_3$ is a full expansion which is an expansion of $M_2$ and hence of $M$.

A *manifold of full expansions* is a manifold, each model of which is a fully expanded model. We now state the *automon eidos* property claimed by Husserl (1970, p. 473, lines 8–23) to hold for definite manifolds:

**Theorem: 20:** A manifold is a Husserl-definite manifold of full expansions iff there exists no manifold of which it is a generalisation.

**Proof:** First, let $\mathfrak{M}$ be any Husserl-definite manifold of full expansions. By lemma 19, no model in $\mathfrak{M}$ has an expansion. Thus there is no manifold $\mathfrak{M}_2$ of which $\mathfrak{M}$ is a generalisation by removals, and by theorem 18, there is no manifold of which $\mathfrak{M}$ is a generalisation.

Secondly, let $\mathfrak{M}\Sigma$ be any manifold which is not a generalisation of any other. By theorem 18, $\mathfrak{M}\Sigma$ is a Husserl-definite manifold. For a *reductio ad absurdum* proof, assume that some model $M$ in $\mathfrak{M}\Sigma$ is not fully expanded. By lemma 19, there exists an expansion $M_2$ of $M$. Let $\Sigma_2$ be the set of all sentences true in $M_2$, and let $\Sigma_1$ be the set of all sentences true in $M$. Since $M_2$ is an expansion of $M$, $\Sigma_1 \subset \Sigma_2$. Moreover, since $M \in \mathfrak{M}\Sigma$, all sentences of $\Sigma$ are true in $M$. I.e. $\Sigma \subseteq \Sigma_1$, and hence $\Sigma \subset \Sigma_2$. In order to show that $\mathfrak{M}\Sigma$ is a generalisation of $\mathfrak{M}\Sigma_2$, it remains only to establish that each model in $\mathfrak{M}\Sigma_2$ is an expansion of some model in $\mathfrak{M}\Sigma$. Let $M_3$ be any model in $\mathfrak{M}\Sigma_2$. Then each sentence in $\Sigma_2$, and hence each sentence in $\Sigma$, is true in $M_3$. Let $M_4$ be the reduct of $M$ whose relations are all and only the denotations in $M_3$ of the predicates of $\Sigma$. It follows that $M_4 \varepsilon \mathfrak{M}\Sigma$. Hence $\mathfrak{M}\Sigma$ is a generalisation of $\mathfrak{M}\Sigma_2$, contradicting the assumption.

**Theorem 21:** Each manifold which contains a countable model is a Husserl-definite manifold of full expansions or is a generalisation of one.

**Proof:** Let $\mathfrak{M}\Sigma$ be any manifold which contains a countable model $M$. Let $M_2$ be a full expansion which is either $M$ itself or any expansion of $M$, and let $\Sigma_2$ be the set of all sentences true in $M_2$. Clearly $\Sigma \subseteq \Sigma_2$. We show that $\Sigma_2$ is categorical by considering cases depending on whether the universe of $M$ is (1) finite or (2) denumerable. In case (1), for each pair of relations in $M_2$, there are predicates $P_1$ and $P_2$ denoting those relations in the language of $\Sigma_2$. There is a sentence of $\Sigma_2$ stating that there are exactly $k$ members of the universe, and indicating exactly which $n$-tuples do and do not satisfy $P_1$ and which $m$-tuples do and do not satisfy $P_2$. Any models in which all of these kinds of sentences are true must be isomorphic to $M_2$. Thus $\Sigma_2$ is categorical.

In case (2), we consider subcases depending on whether there is a relation $R$ in $M_2$ which is a strict well-ordering of the universe $U$ of $M$ (i.e. such that $< U, R>$

is isomorphic to $< N, < >$ where $N$ is the set of natural numbers). If there is such an $R$ in $M_2$, let $P$ denote $R$, and for each natural number $n$, there is a formula uniquely describing the $n^{th}$ element in the ordering $R$. For example, the following formula uniquely describes the third element (in that only the assignment of the third element to $u$ satisfies the formula in $M_2$):

$$\exists\, v\langle \neg \exists\, wPwv \wedge \exists\, v_2 \{Pvv_2 \wedge Pv_2u \wedge \forall w\,[(Pvw \wedge Pwu) \rightarrow (w \approx v_2)]\}\rangle$$

Thus each element of $U$ is uniquely described by a formula in the language of $\Sigma_2$. In the second subcase, there is no such $R$ in $M_2$. Since the set of binary relations of $M_2$ is an ultrafilter, there is a relation $R$ of $M_2$ whose complement $R' = \{ < x, y > \mid x \in U, y \in U,$ and $< x, y, > \not\in R\}$ is a strict well-ordering of $U$. Let $P$ denote $R$. A formula uniquely describing the $n^{th}$ element of $R'$ can be obtained from the one used in the previous subcase by replacing all occurrences of '$P$' with '$\neg P$'. In either subcase, each element of $U$ is uniquely described by a formula in the language of $\Sigma_2$. $\Sigma_2$ contains sentences which specify, using such formulas, any $n$-tuple of members of the universe of $M_2$ (for any $n$) and indicate whether or not that $n$-tuple is a member of any particular $n$-ary relation of $M_2$. $\Sigma_2$ also contains sentences (like axioms 1–6 in the proof of theorem 8) which state that $P$ (or the complement of $P$) is a well-ordering isomorphic to $< N, < >$. *Hence,* every model of $\Sigma_2$ is isomorphic to $M_2$.

Thus in both cases, $\Sigma_2$ is categorical. Hence, by lemma 7, $\mathfrak{M}\Sigma_2$ is Husserl-definite. Also, each model in $\mathfrak{M}\Sigma_2$, being isomorphic to $M_2$, is fully expanded. Since $\Sigma \subseteq \Sigma_2$, $\mathfrak{M}\Sigma$ is either equal to or a generalisation of $\mathfrak{M}\Sigma_2$, which is a Husserl-definite manifold of full expansions.

**Theorem 22:** Each Husserl-definite manifold of full expansions is a formal manifold.

**Proof:** Let $\mathfrak{M}$ be any Husserl definite manifold of full expansions. Let $M$ and $M_2$ be any models in $\mathfrak{M}$, and we must show that they are isomorphic. Let $\Sigma$ be the set of all sentences true in $M$. Then $\mathfrak{M}\Sigma \subseteq \mathfrak{M}$ and by theorem 18, $\mathfrak{M}\Sigma = \mathfrak{M}$. We show, by two cases, that each member of the universe $U$ of $M$ is uniquely described by a formula in the language of $\Sigma$. For the first case we assume that *no* unary relation of $M$ is a singleton. Since the set of unary relations of $M$ is an ultrafilter, the complement of each singleton $\{x\}$, where $x \in U$, is a relation of $M$ denoted by $P_x$ in $\Sigma$. Then $\neg P_x u$ is a formula uniquely describing $x$. In the second case there is a singleton unary relation $\{y\}$ in $M$. For each $x \neq y$, define $P_x$ exactly as in the first case, and define $P_y$ as the predicate denoting $\{y\}$. Clearly, each member of $U$ is uniquely described by a formula in the language of $\Sigma$ in this case as well as in the first case. Let $f$ be the function mapping each member of $U$ to the member of the universe $U_2$ of $M_2$ which satisfies the same formula. Since every model in $\mathfrak{M}$ is fully expanded, each member of $U_2$ must satisfy one of these formulas. Thus the function $f$ maps onto the universe of $M_2$. Moreover, for each $n$, each $n$-tuple is uniquely described by a conjunction of the above kinds of formulas. Thus, for each $n$-tuple and each $n$-ary relation, there is a sentence of $\Sigma$ stating whether or not that $n$-tuple satisfies that relation. These sentences of $\Sigma$ assure that $f$ is an isomorphism. Hence $M$ is isomorphic to $M_2$.

**Theorem 23:** There exists a Husserl-definite manifold containing only one isomorphism type, and any such manifold is coextensive with that isomorphism type.

**Proof:** First, by theorem 8, there exists a categorical set $\Sigma$ of sentences such that $\mathfrak{M}\Sigma$ is a Husserl-definite manifold. Since $\Sigma$ is categorical, $\mathfrak{M}\Sigma$ is an isomorphism type. Since isomorphism types are either equal or disjoint, $\mathfrak{M}\Sigma$ contains only the one isomorphism type, namely $\mathfrak{M}\Sigma$ itself. Secondly, let $\mathfrak{M}$ be any Husserl-definite manifold which contains only one isomorphism type $I$. By theorem 13, $\mathfrak{M} = I$.

**Theorem 24:** There exists a Husserl-definite manifold containing at least two isomorphism types, and the isomorphism types contained as subclasses of any such manifold are not themselves manifolds.

**Proof:** First, by theorem 14 (c), there exists a Husserl-definite manifold $\mathfrak{M}$ which is not a formal manifold. Thus $\mathfrak{M}$ is not an isomorphism type, and by theorem 13, $\mathfrak{M}$ is a union of (at least two) disjoint isomorphism types. Secondly, assume $\mathfrak{M}$ is such a Husserl-definite manifold, and for a *reductio ad absurdum* proof, assume that $\mathfrak{M}_1$ is an isomorphism type contained as a subclass within $\mathfrak{M}$, and that $\mathfrak{M}_1$ is a manifold. Then $\mathfrak{M}$ is a generalisation of $\mathfrak{M}_1$ by weakening axioms, contradicting theorem 18. Thus each isomorphism type contained as a subclass of $\mathfrak{M}$ is not a manifold.

## § 5 Interpreting Whole-Part Notions in Manifold Theory

> Abstraction gets to work on a basis of primary intuitions, and with it a new categorial act-character emerges, in which a new style of objectivity becomes apparant, an objectivity which can *only* become apparent – whether given as 'real' or as 'merely imagined' – in just such a founded act. Naturally I do not here mean abstraction merely in the sense of setting-in-relief of some dependent moment in a sensible object, but ideational abstraction, where no such dependent moment, but its Idea, its Universal, is brought to consciousness, and achieves *actual givenness*. We must presuppose such an act in order that the very sort, to which the manifold single moments 'of one and the same sort' stand opposed, may *itself* come before us, and may come before us *as one and the same* (Husserl, LU VI, § 52).

The intuitive concepts of *whole* and *part* are paradigm cases of the type of concept which, following Husserl, we can characterise as *vague* or (synonymously) *indistinct*. Within the context of this paper we depart in letter (but not in spirit) from Husserl's definition of 'distinctness'. We define a concept as *formally distinct* iff for each two particulars which are (correctly) intended as instances of it, those two particulars are isomorphic. To anticipate our epistemological application of notions developed in § 4, this means that a concept is *completely distinct* if it is the intension of a Husserl-definite manifold of full expansions. Any intuitive concept can be shown to be formally vague or indistinct by showing that there is some sentence true of one but false of some other instance of it. That the intuitive concept of *part* is formally vague in this sense can be shown by a consideration of the Stumpf-Husserl distinction between two distinct types of parts; dependent parts (moments) and independent parts (pieces).

It should be remembered that this distinction was developed originally not within ontology, but within the psychology of perception. Stumpf (1873, pp. 109 ff.) and, following him, Husserl characterised a part as *dependent* iff it cannot be perceived separately from the perceptual whole of which it is a part, and as *independent* iff it can be so perceived. Perceptual examples of countable wholes might be the perception of a minor chord or of a swarm of bees: Each note in the chord and each bee in the swarm is an independent part, while the 'flatted third' sound distinctive of this minor but not of that major chord, or the 'swarm' character distinctive of this group of bees but not that group of birds, are dependent parts of their respective wholes. In his 1929 article on Gestalt theory and the phenomenology of (perceptual) thematics, Gurwitsch (1966, pp. 263–5) reformulated the definition of 'independent part', but maintained the distinction between the two different types of parts. Their mutual insistence on this distinction illustrates the opinion of Stumpf, Husserl, and Gurwitsch that the intuitive concept of *part* is vague in the sense defined above.

Husserl (1970a, Investigation V, § 17) and Gurwitsch (1966, pp. 131–4, 143, 186, 332–49, esp. 340) insist on a similar distinction between two types of whole; a *whole which is perceived* is distinguished from the many instances of the whole which is perceived, each of which is the *whole as perceived*. A single whole which is perceived may be perceived in a variety of ways, i.e. under a variety of aspects or as instancing a variety of types of things. For example, the collection of bees may be perceived as a swarm, as a threat, as an indication of a source of honey, or under any combination of these aspects, and so on. This example provides us with one *whole which is perceived*, but with six *wholes as perceived*. This distinction between two types of perceptual wholes *(wholes which are perceived* vs. *wholes as perceived)* illustrates the formal vagueness of the unqualified concept of *whole*.

We will maintain the distinction between the *whole which is perceived* and the *whole as perceived* in this paper, and will follow Husserl and Gurwitsch in characterising the latter as a dependent part of the former:

Husserl has quite correctly observed that what 'genuinely appears' forms, in the full thing-sense, a *dependent* part which can only possess sense unity and sense independence in a whole necessarily containing empty and indeterminate [i.e. *openly possible*] components (Gurwitsch, 1966, p. 186; Cf. Husserl, 1931, p. 355, lines 7–13, which are inadequately translated; and Husserl, 1973, pp. 96–9).

Since the expression *'whole as perceived'* denotes a dependent part of the *whole which is perceived*, we will use it synonymously with the expression 'thing-tied moment of the whole which is perceived'. In contrast, we define a *moment abstract* as an intension of a class of concrete moments belonging to different wholes which are perceived. In distinguishing thing-tied or concrete moments from moment-abstracta we adopt a distinction proposed by Guido Küng (1967, pp. 172ff) which departs from Husserl's terminology (1970a, pp. 426–32). While dependent parts cannot be perceived independently of their wholes, we claim that they can be so *con*ceived (and that the theme of such a *con*ceiving is a moment-abstractum).

The thing-tied moments of different wholes which are perceived may establish similarities amongst those wholes. For example, all swarms are similar because they are all wholes which are perceived as having thing-tied 'swarm' moments; all minor chords are similar because they are all wholes which are perceived as having thing-tied 'flatted third' moments. The shared character of all 'swarm' (or 'flatted third') moments, i.e. that which all such thing-tied moments have in common, we call a 'swarm' (or 'flatted third') abstractum. We will interpret wholes which are perceived, wholes as perceived (i.e. thing-tied moments of wholes which are perceived), and moment-abstracta (which will also be referred to as 'perceptual types' in § 6) as different structures definable in terms of our manifold theory in this section. Before stating this interpretation, we provide the following comments in order to clarify our motivation.

No whole as perceived is a mere class. The swarm and chord cited as examples might at first consideration appear to be classes, but they are more than mere collections of elements (independent parts, or pieces). They involve not only the founding elements but also their particular *swarm* and *minor tonality* characters. These qualitative characters are relational characteristics of the founding independent parts. The absence of such qualitative and relational co-determinations of the founding independent parts characterises classes which are not wholes as perceived. It is this fact which has motivated our choice of models (rather than sets) to serve for wholes as perceived in the manifold-theoretic interpretation of the present section. Further, it is the relation of wholes as perceived (i.e. of concrete moments) to wholes which are perceived which has motivated us to restrict our definition of 'model' to full expansions and reducts of full expansions. Any reduct is related to each of its full expansions in the same way that a whole as perceived (i.e. a thing-tied moment) is related to each of the possible wholes which are perceivable via it.

Because our primary concern is with abstract, rather than with thing-tied moments, we present no theorems concerning relations amongst thing-tied moments and wholes which are perceived. In this paper we conceive of thing-tied moments (wholes as perceived) as Gurwitsch's *themes of perceptual attention*. He has identified this structure as the nucleus of the perceptual noema, and has characterised the whole which is perceived as a Gestalt contexture of such structures. We accordingly consider the thing-tied moments of a single whole which is perceived to be mutually codetermining and to be unified by Gestalt coherence, and consider the whole which is perceived to be a Gestalt contexture of all its thing-tied moments. Successive thematisations $M_1$, $M_2$, $M_3$ . . . of the *same* whole which is perceived $M$ comprise successive *explications* of the whole $M$. Each explication of $M$ reduces the degree of *material vagueness* expressed as the open possibilities of the inner horizon of the nucleus of the perceptual noema which presents $M$ to attention. When the explication involves *articulating thematisation* (Gurwitsch, 1974, p. 261; Cf. Husserl, 1973, § 50), this specification of the inner horizon is accomplished via the constitution of an individual *Sachverhalt* $< M, M_i >$ in which the whole $M$ is involved as exhibiting one of its thing-tied moments $M_i$. The *telos* of such a process of thematising (or, more specifically, of *explicating immediate concrete moments* of) a single whole which is perceived is the elimination of all open possibilities, and the determination of the entire contexture $M$ of all thing-tied moments $M_i$ of the whole. It is this aspect of Gurwitsch's analysis which motivates our choice of full expansions to serve as possible wholes which are perceived, of some full expansions to serve as actual wholes which are perceived, and of the following manifold theoretic interpretations of the language of whole-part theory.

$M$ is a *whole* iff $M$ is a full expansion or an f.m.r. reduct. $M$ is an *f.m.r. reduct* iff $M$ is a model with finitely many relations. $M$ is a *whole which is perceived* iff $M$ is a full expansion, and $M$ is a *whole as perceived* (also, a *concrete moment* and a *thing-tied moment*) iff $M$ is an f.m.r. reduct. $M$ is a *piece of the whole $M'$* iff $M$ is a whole which is perceived, and the universe of $M$ is a proper subset of the universe of $M'$. The notions *$M$ is the whole $M'$ as perceived, $M$ is an immediate concrete moment of the whole $M'$*, and *$M$ is a thing-tied moment of the whole $M'$* are equivalent; *any* $M$ and $M'$ have this relation iff $M$ is an f.m.r. reduct of $M'$. $M$ is an *immediate part* of $M'$ iff $M$ is a piece or an immediate concrete moment of $M'$.

For any f.m.r. reducts $M$ and $M'$ and their respective classes $\mathfrak{M}_M$ and $\mathfrak{M}_{M'}$ of full expansions, $M$ is *founded on $M'$* iff $\mathfrak{M}_M \subset \mathfrak{M}_{M'}$, and $M$ and $M'$ are *mutually founding* iff neither is founded on the other, but there is some f.m.r. reduct $M_1$ which is founded on both $M$ and $M'$. For each two concrete moments, there is a *founding-founded relation* between them iff one is founded on the other, or they are mutually founding. $M$ and $M'$ are *relatively dependent* iff there is a founding-founded relation between them. $M$ is an *immediate dependent part* of the whole $M_1$ iff $M$ is a part of $M_1$, and for each immediate concrete moment $M'$ of $M_1$ other than $M$, $M$ and $M'$ are relatively dependent. It can be shown that any two distinct immediate concrete moments of the same whole $M$ are relatively dependent, and are thus immediate dependent parts of $M$.

If the whole as perceived $M$ is veridical (non-illusory), the actual whole which is perceived from the point of view of $M$ is exactly one member of $\mathfrak{M}_M$, and if $M$ is non-veridical (presents an illusion to perceptual attention), then the actual whole which is perceived (if there is one) is not a member of $\mathfrak{M}_M$. In this paper we refer on occasion to possible wholes which are perceived from the point of view of $M$ (i.e. to members of $\mathfrak{M}_M$) as 'perceivable wholes', and to $M$ as a 'perceived whole', and define the notion '$M$ is a whole' as '$M$ is a perceived or a perceivable whole'. (We restrict the denotation of 'whole' in this paper to perceived and perceivable wholes; however *eide* (as defined in the next paragraph) may justifiably be conceived as wholes which are neither perceived nor perceivable.)

We refer ontologically to intensions of unions of isomorphism types as *'eide'* and 'universal objects of higher order'. In cases where for some cardinal K, each isomorphism type which is a subclass of the union has some f.m.r. model with a universe of cardinality K as a member, we refer to its *eide* as 'moment-abstracta' and (in § 6) 'perceptual types'. Where the union of isomorphism types is a Husserl-definite manifold of full expansions, we refer to its *eide* as *'automon eide'*, and 'universal *individuals* of higher order'. We define *'an intension* $\pi\mathfrak{M}$ *of some union* $\mathfrak{M}$ *of isomorphism types'* as a hyperultraproduct $\pi\mathfrak{M}$ of $\mathfrak{M}$ (by some ultrafilter on $\mathfrak{M}$ ). This hyperultraproduct turns out to be an entity beyond the class-set theory used in §§ 2–4. Our definition is based on the standard notion of an ultraproduct, as in Kopperman (1972, pp. 74–7), which is a complicated structure built up from a given *set* of models. But the hyperultraproducts we are considering as intensions are built up in an analogous manner from a *proper class* of models.

In taking an ultraproduct of a proper class, the class is treated as a member of 'collections' which are members of the universe of the resulting *hyper*ultraproduct. Thus these collections and the hyperultraproducts themselves are ontologically outside of standard class-set theory, and can be thought of as higher order entities in a realm beyond our class-set theory. In view of our manifold-theoretic interpretation of perceivable wholes as full expansions of some model, of perceived wholes (or concrete moments of perceivable wholes) as reducts of full expansions, and of *eide* as hyperultraproducts of proper classes of models, no perceived or perceivable whole is of an order as high as any universal object of higher order. It does not follow from this fact, however, that all perceived and/or perceivable wholes are of the same order. In stating the following interpretation, we use 'and/or' for weak, and 'or' for strong disjunction:

| **Whole-part notions** | **Manifold-theoretic interpretation:** |
|---|---|
| 1. $M$ is a thing-tied moment of $M'$ ($M'$ is a whole which is perceived as $M$). | 1. $M$ is a model with finitely many relations (an *f.m.r.* model) which is a reduct of the full expansion $M'$. |
| 2. $M'$ is a possible whole which is perceived. | 2. $M'$ is a full expansion. |

| | |
|---|---|
| 3. $M$ is a whole. | 3. $M$ is a full expansion or an f.m.r. model |
| 4. to explicate the whole $M'$ which is perceived. | 4. to expand some f.m.r. model $M$ which is a reduct of the full expansion $M'$. |
| 5. $M$ is described by $\Sigma$. | 5. $M$ is a model of $\Sigma$. |
| 6. $\pi\mathfrak{M}$ is an eidos instanced by $M$. | 6. $\pi\mathfrak{M}$ is a hyperultraproduct (hup.) of a union $\mathfrak{M}$ of isomorphism types such that $M \in \mathfrak{M}$. |
| 7. $\pi\mathfrak{M}$ is a predicative *eidos* instanced by $M$. | 7. $\pi\mathfrak{M}$ is a hup. of some manifold $\mathfrak{M}$ where $M \in \mathfrak{M}$ and $M$ is a model. |
| 8. $\pi\mathfrak{M}$ is an eidos instanced by everything described by $\Sigma$. | 8. $\pi\mathfrak{M}$ is a hup. of $\mathfrak{M}\Sigma$. |
| 9. $\pi\mathfrak{M}$ is an *eidos* which is a moment-abstractum instanced by the thing-tied moment M. | 9. $\pi\mathfrak{M}$ is a hup. of a union $\mathfrak{M}$ of isomorphism types such that $M \in \mathfrak{M}$, and $M$ each $M' \in \mathfrak{M}$ is an f.m.r. model of the same cardinality. |
| 10. $\pi\mathfrak{M}$ is a predicative *eidos* which is an abstract predicative moment-abstractum instanced by the thing-tied moment $M$. | 10. $\pi\mathfrak{M}$ is a hup. of some manifold $\mathfrak{M}$ where $M \in \mathfrak{M}$ and each $M' \in \mathfrak{M}$ is an f.m.r. model of the same cardinality. |
| 11. $\pi\mathfrak{M}$ is a formal *eidos* instanced by $M$. | 11. $\pi\mathfrak{M}$ is a hup. of the isomorphism type of $M$. |
| 12. $\pi\mathfrak{M}$ is a predicative formal *eidos* instanced by $M$. | 12. $\pi\mathfrak{M}$ is a hup. of some manifold $\mathfrak{M}\Sigma$ such that $\Sigma$ is categorical, and $M$ is a model of $\Sigma$. |
| 13. $\pi\mathfrak{M}$ is a formal *eidos* which is a moment-abstractum instanced by the thing-tied moment $M$. | 13. $\pi\mathfrak{M}$ is a hup. of the isomorphism type of $M$ such that $M$ is an f.m.r. model. |

| | |
|---|---|
| 14. $\pi\mathfrak{M}$ is a Husserl-definite *eidos* instanced by $M$. | 14. $\pi\mathfrak{M}$ is a hup. of a Husserl-definite manifold $\mathfrak{M}$ such that $M \in \mathfrak{M}$. |
| 15. $\pi\mathfrak{M}$ is an *automon eidos* instanced by the whole $M$ which is perceived. | 15. $\pi\mathfrak{M}$ is a hup. of some class $\mathfrak{M}$ such that $M \in \mathfrak{M}$ and $\mathfrak{M}$ is a Husserl-definite manifold of full expansions. |
| 16. The *eide* $\pi\mathfrak{M}_1$ and $\pi\mathfrak{M}_2$ are coextensive. | 16. $\pi\mathfrak{M}_1$ is a hup. of $\mathfrak{M}_1$, $\pi\mathfrak{M}_2$ is a hup. of $\mathfrak{M}_2$, and $\mathfrak{M}_1 = \mathfrak{M}_2$. |

Note: $(\pi\mathfrak{M}_1 = \pi\mathfrak{M}_2)$ *implies* $(\mathfrak{M}_1 = \mathfrak{M}_2)$, but *it is not the case that* $(\mathfrak{M}_1 = \mathfrak{M}_2)$ implies $(\pi\mathfrak{M}_1 = \pi\mathfrak{M}_2)$.

| | |
|---|---|
| 17. $\pi\mathfrak{M}$ is a predicative *eidos* which is a least generalisation by weakening axioms of at least two predicative *eide* $\pi\mathfrak{M}_1, \pi\mathfrak{M}_2, \ldots$ | 17. $\pi\mathfrak{M}$ is a hup. of the union of at least two manifolds $\mathfrak{M}_1$, $\mathfrak{M}_2, \ldots$ |

*Whole-part interpretations of the theorems of manifold theory:*

*Note:* The following interpretations are either equivalent to or consequences of the corresponding theorems as stated in §§ 3–4.

| | |
|---|---|
| **Theorem 1:** | The binary relation of two wholes which instance a single formal eidos is an equivalence relation. |
| **Theorem 2:** | No formal *eidos* is a whole. |
| **Theorem 3:** | No Husserl-definite *eidos* is a whole. |
| **Theorem 4:** | No predicative *eidos* is a whole. |
| **Theorem 5:** | Each predicative *eidos* which is coextensive with a formal *eidos* is coextensive with a Husserl-definite *eidos*. |
| **Theorem 6:** | For each axiom set $\Sigma$, if $\Sigma$ is complete, then each predicative *eidos* $\pi\mathfrak{M}\Sigma$ is Husserl-definite. |
| **Lemma 7:** | For each axiom set $\Sigma$, if $\Sigma$ is categorical, then any predicative eidos $\pi\mathfrak{M}\Sigma$ is both Husserl definite and formal. |
| **Theorem 8:** | There exists a categorical set $\Sigma$ of monadic second-order sentences such that each predicative *eidos* $\pi\mathfrak{M}\Sigma$ is Husserl-definite, but $\Sigma$ is not complete. |

**Theorem 9:** For each predicative *eidos* $\pi\mathfrak{M}$, if $\pi\mathfrak{M}$ is Husserl-definite, then there is some perhaps infinite axiom set $\Sigma$ such that $\pi\mathfrak{M}$ is instanced by all and only wholes described by $\Sigma$ and $\Sigma$ is complete.

**Theorem 10:** For each first-order axiom set $\Sigma$, each $\pi\mathfrak{M}\Sigma$ is a first-order Husserl-definite *eidos* instanced by all and only wholes described by $\Sigma$ iff $\Sigma$ is first-order complete.

**Theorem 12:** Each predicative *eidos* is either Husserl-definite, or it is a least generalisation by weakening axioms of at least two extensionally disjoint Husserl-definite *eide*.

**Theorem 13:** Each predicative *eidos* is either coextensive with a formal *eidos*, or is a least generalisation by weakening axioms of extensionally disjoint formal *eide*.

**Theorem 14:** (a) There exists a formal eidos which is not predicative; (b) there exists a predicative eidos which is neither a predicative formal *eidos* nor a least generalisation by weakening axioms of predicative formal eide; (c) there exists a Husserl-definite *eidos* which is neither a predicative formal *eidos* nor a least generalisation of predicative formal *eide*.

**Theorem 15:** Each formal *eidos* $I$ is an *eidos* instanced by something which instances a Husserl-definite *eidos*, but $I$ is not necessarily predicative.

**Theorem 16:** For each formal *eidos* $\pi I$, there exists a Husserl-definite *eidos* $\pi\mathfrak{M}$ such that $\pi I$ is instanced by a whole which instances $\pi\mathfrak{M}$ and every generalisation by weakening axioms of $\pi\mathfrak{M}$.

**Theorem 17:** Each Husserl-definite *eidos* can be generalised by weakening axioms to a predicative *eidos* (which by the next theorem is not Husserl-definite).

**Theorem 18:** Any predicative *eidos* is Husserl-definite iff there exists no predicative *eidos* of which it is a generalisation by weakening axioms.

**Lemma 19:** No whole which is perceived is a whole as perceived.

**Theorem 20:** Any predicative *eidos* is an *automon eidos* iff there exists no predicative *eidos* of which it is a generalisation.

**Theorem 21:** Each predicative *eidos* which is instanced by a countable whole is an *automon eidos* or a generalisation of one.

**Theorem 22:** Each *automon eidos* is a formal predicative *eidos*.

**Theorem 23:** There exists a Husserl-definite predicative *eidos* $\pi\mathfrak{M}$ such that there exists a formal *eidos* $\pi I$ instanced by all wholes which instance $\pi\mathfrak{M}$, and in all such cases $\pi I$ and $\pi\mathfrak{M}$ are coextensive.

**Theorem 24:** There exists a Husserl-definite *eidos* $\pi\mathfrak{M}$ instanced by at least two wholes $M$ and $M'$, where there are no coextensive formal *eide* instanced by both $M$ and $M'$.

## § 6 Ontological and Epistemological Interpretations of Manifold Theory

In this section we discuss some implications of the theory of manifolds from the point of view of the ontological interpretation of § 5, and of the epistemological interpretation (i.e. as a theory of the transcendental constitution of perceptual wholes and universals) which we have had in mind but have not developed in detail.

The problem of carrying out an adequate phenomenology of conceptualisation involves the formulation of descriptions of processes of attention (thematic transitions) within which formalisation, generalisation, and other processes of attention involved in conceptualisation occur. While no such descriptions will be attempted here, certain comments regarding epistemological application of the results of §§ 2–4 and lines of approach to the problem of conceptualisation are appropriate. We begin these comments with a list of epistemological definitions which will enable us to establish the relation of an *Idea in the Kantian sense* to perception, and will conclude by delineating certain types of conceptualisation as problems for further research.

1. A concept is *formally distinct* iff it is an intension of an isomorphism type.
2. A concept is *materially distinct* iff it is an intension of a union of isomorphism types containing as members only full expansions.
3. A concept is completely *distinct* iff it is both formally and materially distinct.

*Remark:* Any intension of an isomorphism type is a formally distinct concept, and any intension of a Husserl-definite manifold of full expansions is a formally and materially (thus completely) distinct concept, and the epistemological correlate of an *automon eidos,* by theorem 22.

4. A concept is *clear* iff it is instanced (in perception) by some thing-tied moment.
5. ($M_f$, M) is an individual Sachverhalt iff $M_f$ is a whole which is perceived, and M is an immediate part of $M_f$.

*Remark:* The existence of an individual Sachverhalt ($M_f$, M) is a necessary and sufficient condition for the truth of a sentence (formulated in some language other than the object languages considered in this paper) ascribing the immediate part M to $M_f$ as an articulated explicate.

6. A process of explicating an immediate dependent part of *a whole $M_f$ which is perceived* is interpreted as a process of expanding $M_1$ to $M_2$ where both $M_1$ and $M_2$ are f.m.r. models which are reducts of the full expansion $M_f$.
7. Each explication of an immediate dependent part of the whole $M_f$ which is perceived via a thing-tied moment $M_1$ results in $M_f$ being perceived via a thing-tied moment $M_2$ (where $M_2$ is a reduct of $M_f$ and $M_1$ is a reduct of $M_2$).

*Remark:* In each such case, the thing-tied moment $M_2$ is an *explicate* of the whole $M_f$ which is perceived.

8. Immediate dependent part explication which involves *articulating thematisation* (see Gurwitsch, 1974, ch. 10; Cf. Husserl, 1973, §§ 22–32, 47–65) constitutes the individual *Sachverhalt* $< M_f, M_2 >$, i.e. the situation of $M_2$ being articulated as a thing-tied moment of the whole $M_f$ which is explicated.

*Remark:* In each such case, the thing-tied moment $M_2$ is an *articulated explicate* of the whole $M_f$ which is perceived.

**Theorem 25:** If a concept $\pi\mathfrak{M}$ is completely distinct, then it is unclear.

**Proof:** If a concept $\pi\mathfrak{M}$ is completely distinct, then it is instanced only by models which are full expansions and isomorphic to each other. Thus $\pi\mathfrak{M}$ is instanced by no

> reduct of any full expansion. By the definition of 'thing-tied moment', $\pi\mathfrak{M}$ is instanced by no thing-tied moment, and is therefore not clear. We note also that if a concept is clear, then it is materially indistinct (materially vague), though it may be formally distinct.

The material vagueness characteristic of perceptual evidence may be progressively diminished by successive explications, and the *telos* of this process (of explicating dependent parts of the same whole which is perceived) is the ideal of *adequate* perceptual (i.e. clear *and* completely distinct) evidence. Perceptual evidence which is *adequate* is therefore never in fact available, but is a limit approached by successive explications of immediate dependent parts of a single whole which is perceived, and is experienced (i.e. is originarily constituted in transcendental consciousness) as the *telos* of such a sequence of explications (Cf. Husserl, 1970a, pp. 720, 731–2, 734–6, 745–8, 760–70). Because of this *limit* feature *vis-à-vis* perceived objects and moment abstracta, a completely distinct concept (i.e. an intension of a Husserl-definite manifold of full expansions) has the status of an *Idea in the Kantian sense* (Cf. Husserl, 1931, §§ 22, 74, 83, 149). The constitution of such a *Kantian Idea* in thematising an *automon eidos* (called by Husserl a *concretum*) is one type of conceptualisation, which we shall refer to as 'ideation'.

We propose that each proper class which is a union of isomorphism types is a unary relation capable of hypostatisation as an *eidos*, i.e. as a more or less general universal (object of higher order) via conceptualisation of one sort or another. Each such proper class of models can be viewed as the extension of an *eidos*. Within our ontological and epistemological interpretations, a hyperultraproduct of such an extension is identified as its intension, and the instantiation of an *eidos* by wholes of a given type is characterised as *instancing*. For any moment-abstractum (perceptual type) $\pi\,\mathfrak{M}_1$ and *eidos* $\pi\,\mathfrak{M}_2$, we define $\pi\,\mathfrak{M}_2$ as *a specification of* $\pi\,\mathfrak{M}_1$ *by explication of immediate dependent parts* iff $\mathfrak{M}_1$ is a generalisation of $\mathfrak{M}_2$ by removals (see p. 458, *supra*). Since the phenomenology of science is concerned with describing an enterprise essentially involving language, we are particularly interested in the cases where the proper classes involved in these processes of conceptualisation are manifolds.

Where the processes of conceptualisation are scientific, the proper classes of models to be hypostatised via abstraction and specified via explication must be expressed linguistically (predicatively). It appears plausible to consider manifolds to be the classes which become hypostatised and specified as universal objects of higher order (predicative *eide*) via theorising activities (i.e. via predicative processes of attention) of various sorts. Our results permit us to distinguish ontologically between those manifolds amenable to hypostatisation as individual universal objects of higher order, and those amenable to hypostatisation as non-individual universals. Theorem 20 indicates that all and only Husserl-definite manifolds of full expansions satisfy the conditions for hypostatisation as universal individuals (i.e. as *automon eide*). Manifolds of f.m.r. reducts can be hypostatised as moment-abstracta which are non-individual universals, i.e. as generalised *eide* which are capable of further specification.

Similarly, we can distinguish epistemologically between distinct and vague concepts on the basis of our manifold-theoretic results. Moment-abstracta are, considered as concepts, materially indistinct. We therefore associate clear concepts with moment-abstracta (perceptual types), considering them as epistemological correlates of generalised *eide*, and contrast them with completely distinct concepts (Kantian Ideas), which we identify as the epistemological correlates of *automon eide*. Since all moment-abstracta are instanced by thing-tied moments while no *automon eidos* is instanced by any concrete moment, no *automon eidos* is a moment-abstractum (perceptual type). However, each *automon eidos* is the *telos* of a sequence of progressively more specific, less general moment-abstracta (perceptual types). This means that ideation should be viewed as involving first the hypostatisation of a moment-abstractum, and secondly the specification of that hypostatised type via thematic transitions (e.g. Husserl's free variation in imagination) which are equivalent to an infinite sequence of progressive specifications which approach the constitution of a distinct concept and the thematisation of an *automon eidos* as an ideal limit of the process of specification. This transition (via specification) from moment-abstractum to *automon eidos* is, we submit, the 'transition from type to (completely distinct) concept and *eidos*' mentioned by Gurwitsch in the passage quoted on p. 442 above. The account of conceptualisation which Gurwitsch envisioned must therefore include not only a description of the processes of attention involved in the hypostatisation of moment-abstracta, but also descriptions of the processes of attention involved in the progressive specification of such hypostatised moment-abstracta. We suggest that such accounts will have to involve descriptions not only of prepredicative and predicative explication in perception (i.e. specification via both *schlichte* and *kategoriale Anschauung;* Husserl, 1970a, pp. 773–815), but also of free variation in the imagination (Husserl, 1973, pp. 321–64 and 1977, pp. 53–78).

The problem of conceptualisation indicated by Gurwitsch as a pressing desideratum of non-egological transcendental phenomenology at the present stage of its development therefore involves more than just indicating the mathematical operation of hyperultraproduct as a promising formalisation of abstraction, and specifiying which relations can, and which cannot be thematised via abstraction as universal individuals and/or objects of higher order. It requires also epistemological descriptions (to be developed in terms of the general theory of intentionality) of the processes of attention which are involved in such abstractive hypostatisations, as well as of other processes of attention (such as specification to less general moment-abstracta, and ideation to *automon eide*). In view of the distinctions we have developed in this paper, we suggest that the processes of attention involved in the constitution of generalised and *automon eide* (and the corresponding epistemological structures which we have called concepts of various sorts) should be described separately and in relation to each other. We further suggest that this epistemological task can be approached in terms of the notion of hyperultraproduct and manifold theory, and that it ultimately cannot avoid the concept of the lifeworld.

To this end, we propose considering (*ex hypotheosi*) *the theme of perceptual attention* (i.e. the nucleus of the perceptual noema) to be an f.m.r. model of some axiom set $\Sigma$, the *order of existence* O of $M$ to be the class of f.m.r. models which are cardinally equivalent to $M$ and compatible with the language of $\Sigma$, the *thematic field* $T$ of $M$ to be $\mathfrak{M}\Sigma \cap \mathrm{O}$, and the *principle of material relevance* which organizes $T$ to be the moment abstractum $\pi T$. Some relevant processes of conceptualization would then be:

*Ontological abstraction*: Those thematic transitions by which attention can, given as an initial theme some whole $M$ perceived as a member of some thematic field $T$, constitute as a subsequent theme the *eidos* (moment abstractum, or perceptual type) $\pi T$.

*Specification by strengthening axioms*: Those thematic transitions by which attention can, given as an initial theme some whole $M$ perceived as a member of some thematic field $T$, constitute as a subsequent theme the whole $M$ perceived as a member of a thematic field $T'$ such that $\mathfrak{M}\Sigma$ is a generalisation of $\mathfrak{M}\Sigma'$ by weakening axioms.

*Specification by explication of immediate dependent parts*: Those thematic transitions by which attention can, given as an initial theme some whole $M$ perceived as a member of some thematic field $T$, constitute as a subsequent theme the whole $M$ perceived as a member of some thematic field $T'$ such that $M$ is an immediate dependent part of $M'$ and $\mathfrak{M}\Sigma$ is a generalisation of $\mathfrak{M}\Sigma'$ by removals.

*Specification*: Those thematic transitions by which attention can, given as an initial theme some whole $M$ perceived as a member of some thematic field $T$, constitute as a subsequent theme the whole $M'$ perceived as a member of a thematic field $T'$ such that $\mathfrak{M}\Sigma$ is a generalisation of $\mathfrak{M}\Sigma'$, and either $M = M'$, or $M$ is an immediate dependent part of $M'$.

*Formalising abstraction:* Any combination of ontological abstractions and/or specifications whereby a formal *eidos* is thematised.

*Ideation*: Any combination of ontological abstractions and/or specifications whereby an *autonom eidos* is thematised.

In conclusion, it should be noted that specification is the inverse of generalisation, and that we have indicated the need for an epistemological account of specification rather than of generalisation because ideation is a limit approached by progressive specification, rather than by generalisation. It should also be noted that we have made no attempt in this paper to formulate epistemological descriptions of the processes of attention defined here as involved in conceptualisation, but have merely categorised these processes in terms of the types of concepts they involve, and the types of universal objects which they constitute as themes of attention. While we have suggested the mathematical operation of ultraproduct as defined on unions of isomorphism types as the structure we consider interpretable as ontological abstraction, we have offered neither a mathematical definition, nor an epistemological interpretation (i.e. description in terms of the general theory of intentionality) of that operation. Such a definition, epistemological descriptions, and formal ontological specifications of the properties common to each life-world as experienced (i.e. to each cultural world), and consideration of the role of culture and consensus in determining the properties characteristic of a particular cultural world then remain outstanding desiderata within the programme sketched by Gurwitsch. Hence the results which we have presented here comprise prolegomena to further work in that area of metaphysics indicated by Gurwitsch as the *problem of conceptualisation*.

## Bibliography


Carr, David, 1974, *Phenomenology and the Problem of History*, Evanston: Northwestern University Press.

Enderton, Herbert B., 1972, *A Mathematical Introduction to Logic*, New York: Academic Press.

Gödel, Kurt, 1930, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", *Monatshefte für Mathematik und Physik*, 37, 349–60. English translation in van Heijenoort, 1967.

– 1931, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme", *ibid.*, 38, 173–98. English translation in van Heijenoort, 1967.

Gurwitsch, Aron, 1964, *The Field of Consciousness*, Pittsburgh: Duquesne University Press.

– 1966, *Studies in Phenomenology and Psychology*, Evanston: Northwestern University Press.

– 1974, *Phenomenology and the Theory of Science*, ed. by Lester Embree, Evanston: Northwestern University Press.

– 1979, *Human Encounters in the Social World*, trans. by Fred Kersten, Pittsburgh: Duquesne University Press.

Husserl, Edmund, 1891, *Philosophie der Arithmetik*, as repr. in Husserl, 1970.

– 1931, *Ideas*, trans. by W. R. Boyce Gibson, New York: Humanities Press. (Orig. publ. 1913.)

– 1969, *Formal and Transcendental Logic*, trans. by Dorion Cairns, Haag: Nijhoff. (Orig. publ. 1929.)

– 1970, *Philosophie der Arithmetik mit ergänzenden Texten (1890–1901)*, ed. by Lothar Eley, Husserliana XII, Haag: Nijhoff.

– 1970a, *Logical Investigations*, trans. by J. N. Findlay, 2 vols., New York: Humanities Press. (Orig. publ. 1900/01, 2nd ed. 1913/21.)

– 1970b, *The Crisis of European Science and Transcendental Phenomenology*, trans. by David Carr, Evanston: Northwestern University Press. (Orig. publ. 1938.)

– 1973, *Experience and Judgment*, trans. by James S. Churchill and Karl Ameriks, Evanston: Northwestern University Press. (Orig. publ. 1938.)

– 1977, *Phenomenological Psychology*, translated by John Scanlon, Haag: Nijhoff. (Orig. publ. 1925.)

Kopperman, Ralph, 1972, *Model Theory and Its Applications*, Boston: Allyn and Bacon.

Küng, Guido, 1967, *Ontology and the Logistic Analysis of Language*, New York: Humanities Press.

Monk, Donald J., 1969, *Introduction to Set Theory*, New York: McGraw-Hill.

Null, Gilbert T. and Simons, Roger A., 1981, "Aron Gurwitsch's Ordinal Foundation of Mathematics and the Problem of Formalizing Ideational Abstraction", *Journal of the British Society for Phenomenology*, 12, 164–74.

Russell, B., and Whitehead, A. N., 1910/13, *Principia Mathematica*, Cambridge: Cambridge University Press.

Schutz, Alfred, 1964, *Collected Papers*, 3 vols., Haag: Nijhoff.

Schutz, A. and Parsons, T., 1978, *The Theory of Social Action*, ed. by Richard Grathoff, Bloomington and London: Indiana University Press.

Stumpf, C., 1873, *Über den psychologischen Ursprung der Raumvorstellung*, Leipzig: Hirzel.

– 1883/90, *Tonpsychologie*, 2 vols., Leipzig: Hirzel.

Tarski, Alfred, 1956, "The Concept of Truth in Formalized Languages", in *Logic, Semantics, Metamathematics*, Oxford: Clarendon Press.

Tugendhat, Ernst, 1967, *Der Wahrheitsbegriff bei Husserl und Heidegger*, Berlin: de Gruyter.

Van Heijenoort, Jean, 1967, *From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931*, Cambridge, Mass.: Harvard University Press.

Wiener, N., 1912/14, "A Simplification of the Logic of Relations", *Proc. Camb. Phil. Soc.*, 17, 387–90, repr. in van Heijenoort, pp. 224–27.

Wilder, Raymond L., 1952, *Introduction to the Foundations of Mathematics*, 2nd ed., New York: John Wiley & Sons.