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Three Essays in Formal Ontology

Prefatory Note

When first reading Husserl's *Logical Investigations* it is very easy to pass by the third as a minor detour from the high road of Husserl's major concerns. In common with many other readers, I initially held this view: the many distinctions Husserl makes seemed to me to be, to use his own words about Twardowski, 'as subtle as they are queer'. To anyone accustomed only to the extensional whole-part theories of Leśniewski or Goodman this is a natural reaction. My change of view was influenced partly by Kevin Mulligan's insistence on the pivotal role of the third investigation in Husserl's work, and also by the increasing recognition of the themes of unity, dependence and self-sufficiency treated by Husserl, as concepts echoing loudly throughout the history of ontology. It was also Kevin Mulligan who unearthed Ginsberg's 1929 article on Husserl's six theorems, and discontent with her criticisms spurred me to attempt a formalised reconstruction of Husserl's ideas, which met with various difficulties on the way to the first of these essays.

At the same time I was attempting to use mereological considerations to offer an alternative to what I consider the unacceptable account of number put forward by Frege, using Schröderian and Husserlian ideas suggested to me by Barry Smith. My original view was that numbers are properties of what I then called manifolds, i.e. aggregates considered as composed in some determinate way. This is what I should now call a group or aggregate theory of number. In the second essay I present the considerations which forced me to abandon such a view and to recognise the distinctive nature of pluralities as against aggregated individuals. This in turn led me to reappraise the notions of reference and set, with the result seen in the third essay, where a formal theory of mani-

fold, now reconstrued as comprising both individuals and pluralities, is developed. Some manifolds are aggregable: to such aggregates mereological considerations still apply. These issues are dealt with in the second essay, where the opposition to Frege is also explicitly set out.

At each turn I found voices of encouragement from the past, some from unexpected quarters. Harkening to these has convinced me that the logical and philosophical harvest of the fecund years between Husserl's *Philosophie der Arithmetik* in 1891 and Russell's *Principles of Mathematics* in 1903 is yet far from being reaped in full.

I. The Formalisation of Husserl's Theory of Wholes and Parts¹

§ 1 Introduction

Husserl's third Logical Investigation is called "*On the theory of wholes and parts*". It has probably received less attention from commentators than any of the other investigations, including the shorter fourth, which Husserl himself saw as an application of the ideas of the third to questions of grammar. The ideas put forward in the third investigation play a crucial role in Husserl's subsequent philosophy, and he was able to recommend them, even much later in his life, as offering the best way into his philosophy.² Although they did not perhaps present such an attractive clarion-call to research, they might, had Husserl's advice been followed, have made a much greater contribution to philosophical work than in fact they did. I should like to suggest that it is not too late to learn from the third investigation, and that, in a tidier form than they there receive, the ideas could become indispensable weapons in the conceptual armoury of the philosopher interested in ontology. This paper has the more modest purpose of attempting to clarify and interpret what Husserl was trying to say, with a view to eventually offering a rigorous treatment of the most important notions, and I wish also briefly to suggest where such notions might prove important in ontology.

It is important to distinguish formalisation from mere symbolisation. Any expression may be symbolised: one simply introduces symbols for various words or other expressions: the difference is merely one of the graphic shape of the expression. However, symbols, unlike the natural language expressions they can conventionally replace, derive their sense from the specific convention setting up their use, whereas this freedom of interpretation is not available for the original natural language expressions. For this reason symbols are more easily detachable from their specific interpretation, and may be manipulated purely syntactically, without interpretation. It is this feature which makes symbolisation such a useful way of presenting a formal theory. A formal theory, in Husserl's sense,³ is one in which no mention is made of any particular things or kinds of things, but which deals with objects in complete ab-

straction from their specific natures. A formal theory need not even be expressed symbolically: a statement such as, 'If a thing bears a relation to another thing, then the second thing bears the converse relation to the first',⁴ contains no restriction to particular domains of application, but consists purely of logical constants and formal concepts, such as *thing*, *relation*, and concepts such as *converse* definable in terms of these. It is advantageous to present formal theories symbolically because we may use symbols which are not given any fixed interpretation, but belong to a grammatical class which corresponds to a formal concept; they are then free to vary in interpretation over any entities whatever falling under that formal concept. So, if we allow the usual sorts of formal grammar, the above formal statement, could be symbolised (If aRb then $b\check{R}a$). Symbolisation usually proceeds further, with symbolisation of the logical constants, which may indeed be necessary if they need some degree of regimentation for the specific purpose in hand. In this sense, a symbolised presentation of a purely formal theory in Husserl's sense fulfils the conditions suggested by Wittgenstein as marking an adequate *Begriffsschrift*.⁵ Each formal concept corresponds to a different type of variable, i. e. symbol with variable interpretation. Only the logical constants are fixed.⁶

Husserl thought that a purely formal theory of part and whole was possible, and regarded the second part of the third investigation as offering the beginnings of such a theory. But, for all its detail, the investigation remains only a sketch of what a fully developed formal theory would look like, and like all philosophical sketches, presents problems of interpretation, lacunae, and vagueness, as well as being highly suggestive of possible fruitful developments. Although Husserl makes a brief and somewhat half-hearted venture into a partial symbolisation of a few theorems, the investigation is largely couched in Husserl's semi-technical German, and he nowhere attempts to set up a formal language in the modern logistic sense, which means that his formal treatment falls well short of modern standards in terms of the rigour of its symbolisation. While Husserl was by no means unfamiliar with symbolic logic as such, he was less interested in symbolisation for its own sake than in the philosophical treatment of concepts, even those concepts where, as in logic and mathematics, symbolisation had become indispensable to progress. He never believed that problems could be resolved purely by recourse to symbolisation, and rejected strongly formalist tendencies in mathematics, which would have us believe that mathematics is simply a

game with symbols which do not themselves have any meaning.⁷ It might be suggested that a theory of whole and part cannot be formal in Husserl's sense, since where – as in the work of Leśniewski and Goodman – it has been formalised hitherto, it has proved to be a proper extension of logic in the normally accepted sense.⁸ Against this it must be pointed out that Husserl clearly states that *whole* and *part* are purely formal concepts.⁹ Whether Husserl is correct on this, depends on what is taken as the criterion for being a formal concept. I do not believe that enough has as yet been done in clarifying the idea of a formal concept to give a definitive answer on this point. To that extent, the title of this essay promises something which it is not clear can be given. However, to the extent that we can eliminate from the theory all *other* concepts which are clearly not formal, to that extent we have succeeded in outlining what Husserl would call a theory of the pure forms of whole and part.¹⁰

Although advertised as a theory of whole and part, Husserl's investigation spends as much time on the concepts of dependence and independence, which, while they bear crucially on Husserl's particular brand of whole-part theory, cannot be counted as purely mereological notions. However, Husserl lays great stress on the distinction between dependent and independent parts as being the chief distinction among parts, and since it is in this distinction that Husserl's theory is distinguished from later and symbolically more adequate whole-part theories, I shall also consider the question of dependence and independence in some detail.

Husserl draws a distinction in the investigation between two different kinds of part or constituent of a whole. Some parts, those normally so-called, could exist alone, detached from the whole of which they happen to be part. These Husserl calls 'pieces' or 'independent parts' of the whole. On the other hand there are parts or constituents of a whole which could not exist apart from the whole or sort of whole of which they are part. These Husserl calls 'moments' or 'dependent parts' of the whole. For example: the board which makes up the top of a table is a piece of the table, while the surface of the table, or its particular individual colour-aspect, are moments of it. This distinction amongst kinds of parts is certainly not new: indeed it may be claimed to go back to the *Categories* of Aristotle.¹¹ Husserl himself certainly derived the distinction from his teacher Stumpf, who used the terms 'partial content' and 'independent content', in his discussion of the distinction within the realm of phenomenological psychology.¹² Husserl first used the distinc-

tion himself in his 1894 article "Psychological studies in elementary logic", where many of the distinctions later made in the *Logical Investigations* are already to be found.¹³ The later exposition contains two major advances on the earlier version: firstly a recognition that the dependent/independent distinction has application outside the sphere of psychological contents to ontology generally, and secondly, connected with this, the idea of a formal theory of whole and part, which, as we have said, Husserl sketches but does not completely execute in the second half of the investigation.

In the hands of later whole-part theorists such as Leśniewski and Goodman, whole-part theory has become associated with nominalism and extensionalism, where its general applicability and algebraic similarities with set theory make it a substitute for set theory more acceptable to those who have ontological objections to sets as abstract objects.¹⁴ Part of the interest in examining Husserl's whole-part theory is that it is free from such nominalist scruples, being conceived within the richly Platonist ontology of pure species adopted by Husserl at the time. It is, furthermore, non-extensional, making indispensable use of the concepts of essence and necessity. The basic distinction Husserl makes between dependent and independent parts is not even expressible in an extensional language. However it seems to me that one need not buy Husserl's package of Platonism and non-extensional language as a whole in order to make use of his whole-part theory. It is usually taken for granted that a non-extensional language brings ontological commitment to Platonic entities of some kind, whether species, meanings, or something like possible worlds. But it is far from clear that we can even manage to make reasonable sense of the actual world in a purely extensional language. It may, further, be possible to use a whole-part theory of Husserl's type to buttress a more sophisticated nominalistic approach to universals *via* Husserlian moments, so the usual yoking together of Platonism and non-extensionalism is far from clearly established.

One of the problems with the interpretation of the third investigation is that not all traces of Husserl's earlier psychological approach and interests have been expunged. This affects both the language within which Husserl makes his points, and the range of examples to which he generally makes recourse. Thus the word 'content' is frequently used where the word 'object' is also appropriate, and where the latter ought to be used in preference. This is despite Husserl's acceptance that his remarks hold for all objects generally, and not just psychological contents. The

examples are drawn almost exclusively from the phenomenological psychology of perception; for instance, that in the visual perception of a coloured thing, the moment of colour and the moment of spatial extension are both dependent parts of the thing as a whole, and require each other's co-occurrence in the thing. When this observation is transposed from the phenomenological to the ontological mode, this yields the proposition that the moment of colour and moment of extension of the thing itself (rather than the thing as perceived) are dependent parts of it. In this case the transposition seems to go quite smoothly, and I believe that it was Husserl's opinion that this would be so quite generally: for 'content' substitute 'object' and the theory has been in principle extended. It seems to me questionable whether the extension of the theory to objects in general is in fact so easy. Particular attention must be paid to the fact that some objects at least may belong to more than one kind at once, and that its dependence relations vis-à-vis other objects may vary according to the kind. This consideration is lacking from the psychological case, and so may have been at work in moulding Husserl's thoughts about the general properties of the more important part-whole relations. It is often difficult to tell, at crucial junctures in the text, whether the unthematized background of examples was playing a part, and if so, what part.¹⁵

Arising from this is the fact that it is in general possible to give the concepts of dependence and independence a much wider application outside the theory of whole and part. Husserl may not have been unaware of this, but he does not embark on any such general development. I have therefore allowed myself to go beyond the range of Husserl's examples in order to open up the question of such a generalised theory of dependence. The attendant risks of distortion and misrepresentation of Husserl's own position are I believe worth running if we are to put his ideas to work quite generally.

§ 2 Problems of Formalisation

There is a wide range of formal languages among which to choose when we attempt to formalise Husserl's ideas. Choice among these must be motivated by considerations partly external to whole-part theory as such. But whichever language is chosen, it cannot, if it is to do justice to Husserl's ideas, be extensional. The whole-part theories of Leśniewski

on the one hand and Leonard and Goodman on the other are both extensional. So a minimalist solution to the choice problem would be to add to one of these a necessity operator and axioms for it. One could for instance take the axioms and rules of S4 and graft these on to the Leonard-Goodman calculus of individuals. This approach has all the merits of timidity: it causes least disturbance. But there are drawbacks as well. Since Husserl was writing before it was appreciated how modal logic would proliferate different systems, there is no chance of receiving a direct answer from his writings as to which of the many available would be the best to choose. In view however of formula (3) below, which tells us that whenever species stand in a relation of foundation they do so of necessity, it appears that any modal system used would have to contain the characteristic S4 axiom $\Box p \supset \Box \Box p$ as a theorem. One obvious candidate modal system is accordingly S4. However, since the applications of modal considerations in the present context do not seem to require that we decide among alternatives whose differences do not show up in the sorts of formula we shall be considering, I shall in fact shirk the choice, and suggest merely that the modal axioms be not weaker than S4.

There is a problem about using a propositional necessity operator at all, in that traditionally the term 'essence' has related not to propositions but to properties, to *de re* rather than *de dicto* necessity. Husserl's writings show a willingness to accept both that individuals of certain kinds possess essential properties, and that there are general essences or *eide*, which are the abstract objects of imaginative variation among possibilities. For this reason I suggest that in addition to a necessity operator on propositions it is advantageous to consider a necessity operator on predicates, or property-abstracts. I shall use the expression 'nec' for this purpose. The operator was introduced by David Wiggins,¹⁶ who has given strong reasons, independent of Husserlian considerations, for believing that such an operator is indispensable to our ordinary conceptual scheme. It remains to be seen how 'nec' and ' \Box ' should be taken to interact, indeed whether a unified theory of them is possible at all. Because of these uncertainties, the account given in this paper must be regarded as only a tentative investigation into essentialistic whole-part theory.

There is yet a further reason for disquiet over simply grafting modal operators onto extensional mereology. For in extensional mereology (which I take to comprise both Leśniewski's mereology and the Leonard-Goodman calculus of individuals) a thing is identified with the

sum of its parts; indeed Goodman *defines* the identity of things as consisting in their having the same parts. But this rules out in advance the possibility of different things merely *coinciding* spatio-temporally. The case where such coincidence does not extend throughout the total life-span of both things is usually handled within extensional mereology by reconstruing things as four-dimensional space-time worms, and pointing out that temporary coincidence merely involves two such entities overlapping in a certain spatio-temporal region. However, there may also be cases in which we should wish to say that two things coincided over their total life-span, yet were not identical. This is connected with the fact that according to the everyday notion of a material thing, a thing can both gain and lose parts without prejudice to its identity, as can, most obviously, an organism.¹⁷ But a whole which conforms to the sum-principle of extensional mereology cannot lose any part. One way of avoiding recourse to four-dimensional objects, but which preserves the sum-principle, is Roderick Chisholm's theory of *entia successiva*.¹⁸ However, it seems somewhat drastic to abandon the paradigmatic role of organisms among material individuals for the sake of an abstract principle, when the normal three-dimensional thing-concept has not conclusively been shown to be beyond redemption. It would further be premature to abandon the normal conception in expounding Husserl's whole-part theory, if there is, as I believe, a chance that this very theory could provide assistance in explicating the normal conception of a thing.¹⁹

So I shall not be following a minimalist line: our mereology will not have the principle that coincident things are identical, and we shall use a *de re* necessity operator. It follows that the suggestions contained in this paper are largely exploratory: like Husserl's this is not a formal presentation with axioms and theorems, but an attempt to set out some of the possibilities and clarify some of the issues which need to be resolved before a formalisation of Husserl's ideas which is both intuitively and formally adequate can be presented.

One respect in which Husserl's whole-part theory is distinctive is its essential use of what Husserl calls *pure species*. I shall use lower-case Greek letters α , β etc. for such species, and lower-case Italic letters *a*, *b*, *c*, etc. for arbitrary members of α , β , γ respectively. Where we are treating an individual as such, in abstraction, as far as possible, from considerations of which species it belongs to, I shall use the letters *s*, *t*. Expressions of the form '*s* \in α ' will mean '*s* belongs to the species α '. But there

is here a problem of interpretation. What are such species? Do they indeed exist? If we follow Husserl in assuming that they do, we run the risk of building too many ontological presuppositions into the formalisation in advance. I shall accordingly give expressions of the form ' $s \in \alpha$ ' as far as possible a merely syntactic reading, allowing ' α ' to replace a common noun, and reading it as ' s is an α '. This leaves it open until later whether we should treat α as a proper name of a pure species, or of a set, or merely as a common name for s and maybe various other individuals. One thing to note, however, if we are to remain faithful to Husserl's way of construing species, is that we cannot allow contradictory species. Every species is, for Husserl, such that it *could* have members, even if it in fact does not. We shall accordingly make the informal stipulation that substituends for α , β etc. should be such that ' $\diamond (\exists x) (x \in \alpha)$ ' should be true.²⁰

Husserl explicitly warns the reader that he is using the term 'part' in a wider sense than it is usually given. He wishes it to comprise not only detachable pieces but also anything else discernible in an object, anything that is an actual constituent of it, apart from relational characteristics.²¹ In Aristotelian terminology, Husserl's parts would comprise parts normally so-called, accidents, and also boundaries.²² Doubts about the propriety of such a treatment are expressed by Findlay in the introduction to his English translation of the *Logical Investigations*. Findlay suggests that while there may be analogies between parts in the usual sense and individual accidents or moments, the two do not belong to the same category and it is therefore a mistake to treat them together ontologically. This does not recognise the expressly formal nature of Husserl's theory, for it is precisely the independence of restrictions to any particular category or region which mark what Husserl calls a formal theory. Husserl's account proceeds independently of doctrines concerning categories and category-mistakes.²³ The only way in which Husserl could be, in his own terms, mistaken, would be if he had confused either two formal concepts, or one formal concept and one material. Given only that Husserl does believe in individual accidents or property-instances, he cannot but treat them as falling within the formal concept of part. It is true that many philosophers have disputed whether there are such accidents. In answer it can be pointed out that not all the examples Husserl adduces as moments are property-instances; there are also boundaries, although he did not expressly include the latter until the later work *Experience and Judgment*.²⁴ It would be uncharitable to expect Husserl to pro-

duce a justification for treating of moments along with other parts in advance of judging how well the theory so produced managed to solve problems of unity and predication by comparison with other competing theories.

§ 3 Husserl's Basic Concepts: Whole and Foundation

The two most important concepts employed by Husserl in the third investigation are those of *whole* and *foundation*. Unfortunately, both these terms are ambiguous, and we must recognise their various senses before we can make clear sense of Husserl's theory. By contrast, Husserl does not make thematic the marks of the general concept *part* as such, but proceeds rather to make distinctions among the various kinds of part. It must be assumed that he considers the concept too primitive, being a formal concept, to allow of substantive elucidation.

Husserl distinguishes three different concepts of whole, a *narrow* concept, a *wide* concept, and a *pregnant* concept. The first two terms are mine; the last is Husserl's own. It is characteristic of Husserl's approach in the *Investigations* that he is reluctant to coin special terminology, even where he recognises ambiguities and is attempting to avoid them. This is in contrast to his later willingness to develop a specifically phenomenological vocabulary.

A narrow whole is one in which a number of entities are bound together into a unit by a further entity which Husserl calls a 'unifying moment' (*Einheitsmoment*). Narrow wholes are a rather special kind of whole, and cannot comprise all the wholes that there are. The supposition that all wholes are narrow in this sense leads, as Husserl points out in a passage reminiscent of Bradley,²⁵ to every complex being, appearances notwithstanding, infinitely complex. For if A and B are bound together by U, then A and U must be bound together by U₁, and so on *ad infinitum*. Husserl's own theory offers a way out of this regress of parts, by suggesting that some kinds of entity come together to form wholes just because they are the kinds of entity that they are, and thereby require partners, without requiring anything else which joins them together.²⁶

The wide concept of whole seems to me to be very like Goodman's concept of an individual; no restrictions are placed on how tightly or loosely connected the various parts of the whole are, whether they are

scattered or not, so long as we can still regard the whole as a single thing. It is indeed in the possibility of being regarded as a single thing that Husserl considers that bare unity consists.²⁷ This does not mean however that there are only individuals. Husserl expressly contrasts *unity* and *plurality* as formal concepts.²⁸ But any plurality may be taken together as something unitary, thereby founding a new higher unity, whose unity is, however, extrinsic to it, in the collective act.²⁹ So I shall allow as individuals anything which can possess a (singular) proper name. This will include even arbitrary *collectiva*. This liberality is reflected in extensional mereologies by allowing that arbitrary sums of individuals are themselves individuals. The reason for this is not that we wish to take most of these arbitrary *collectiva* seriously, but rather that it is not clear in advance where to draw the line between things which are wholes in this widest and weakest sense, and those which have some more intrinsic unity.³⁰

The third or pregnant concept of whole is defined by Husserl in terms of the concept of foundation. A pregnant whole is one each of whose parts is foundationally connected, directly or indirectly, with every other, and no part of the whole so formed is founded on anything else outside the whole. This of course presupposes Husserl's own concept of foundation, which means that Husserl attempts to define one sense of *whole* in terms of foundation, which in fact itself presupposes another concept of whole, the wide concept, which is, as Husserl points out, not a real or determining predicate.³¹ The unity of a pregnant whole is intrinsic to it, by contrast with the extrinsic unity of a mere sum or aggregate.

When we turn to foundation, matters are not so clear. It is most important to clarify Husserl's meaning here, since the concept of foundation turns out to be the most basic one of the whole investigation. I believe that we must distinguish two very different types of relation, both of which Husserl calls 'foundation'. There is a *generic* concept, which relates species, and there is an *individual* concept, which relates individuals which belong to species related according to the generic relation. It would in fact be more correct to speak of generic and individual concepts in the plural, since Husserl offers several formulations which do not exactly coincide, and it is possible to discern further definitional possibilities not considered by Husserl. It is chiefly in connection with the generic relations that one can speak, as Husserl does, of laws of essence. Husserl is mainly interested in the essential relations, and so does not offer an account of individual relations as such. But if one is to be

able to discuss the foundational relations of determinate individuals, such an account is needed, and there are crucial places in the investigation where Husserl is clearly talking about relations between determinate individuals, albeit individuals considered as belonging to a certain species. In his official introduction of the concept of foundation, Husserl, in addition to speaking of the case where the species α and β are foundationally related, also mentions the case where we should say that two members a and b of these respective species are themselves foundationally related.³² The definition of relative dependence and independence offered earlier speaks clearly of one thing's being dependent or independent relative to another.³³ In each case it is clear from either the context or the notation that the schematic letters used by Husserl are to be taken as singular terms.³⁴ Finally the notion of a pregnant whole requires that we talk about the foundational connectedness of the individual parts making up the whole. For these reasons an account of the foundational relatedness of individuals is necessary. However, Husserl was of the opinion that it is possible to move back and forth between talk about individuals and talk about species without difficulty, and so does not enlarge upon the difference.³⁵ It is however this difference which constitutes the major difficulty in developing a Husserlian whole-part theory.

Husserl defines foundation in the first instance as a relation holding between two pure species. The verbal rendering of the definition goes thus:³⁶

an α as such requires	foundation by a β	:=	there is an essential law to the effect that an α cannot exist as such except in a more comprehensive unity which associates it with a β .
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Later Husserl contends that the concept of whole or unity here employed is dispensable, and reformulates what he takes to be the same idea thus:³⁷ in virtue of the essential nature of an α , an α cannot exist as such unless a β also exists.

In this second version reference to a more comprehensive whole is missing. But this suffices to make the two concepts of foundation not equivalent. For according to the second definition, every species is self-founding. This means that, according to a statement Husserl elsewhere makes about absolute dependence,³⁸ everything is dependent absolutely. This is clearly not what Husserl wanted since it obliterates the distinc-

tion between dependent and independent objects. So the concept of foundation used in defining absolute and relative dependence cannot be the weaker second concept. One solution to this problem which readily suggests itself is that the species α and β have to be *different*. We could then say that something is dependent only if it is dependent on something belonging to another species.³⁹ This will not do, however, since it turns out that there are species which are non-trivially self-founding, which the suggestion does not allow.

So we shall revert to Husserl's first formulation, with its reference to a more comprehensive unity. This suggests that every α should be found together with a β in something of which the α in question is a *proper* part. In what follows, we shall use Goodman's symbols ' \ll ' for 'is a proper part of' and '<' for 'is a part of', where the latter allows, while the former excludes, coincidence. Hence our suggestion for a rendering of the definition is:

$$(1) \quad \square(\forall x)(x \in \alpha \supset (\exists yz)(y \in \beta \ \& \ x \ll z \ \& \ y < z))$$

This condition appears still not to be strong enough. Whilst it captures the letter of Husserl's formulation it misses something of the spirit, in that in line with this definition the more comprehensive whole could be simply the β itself. This appears implausible as capturing the idea that we want: while we might say that the species *husband* is founded on that of *wife* (and vice versa of course) we should not want to say that because the existence of husbands required that of married couples that husbands are founded as such on married couples. This appears to have got marital carts before horses. Similar remarks would apply, *mutatis mutandis*, to foundation relations cited by Husserl, such as the mutual foundedness of colour-moments and moments of extension. Husserl takes foundation to be a relation of necessary *association*,⁴⁰ and the connotations of this word preclude either the α or the β in question from exhausting the more comprehensive whole of which each is a part. Indeed Husserl's formulation is itself ambiguous in that it could be read as implying that the whole is more comprehensive just than the α or as more comprehensive than both the α and the β , which is our second and preferred reading. Although Husserl does talk of wholes in the pregnant sense being founded upon the range of their parts,⁴¹ this is a regrettable equivocation, and probably stems from the etymology and previous use by others as well as Husserl of the word. In the sense we have formulat-

ed, everything which is founded on something is thereby dependent, whereas in this other sense we could describe even independent wholes as founded upon their parts. It would be better to describe such wholes as *constituted by* their parts, reserving the word 'foundation', despite its misleading etymology, for the associative relationship. However, we cannot merely strengthen the last conjunct ' $y < z$ ' of (1) to ' $y \ll z$ ', for it would follow from this that any species whose instances had to exist as part of some greater whole would thereby be a self-founding species. But while a lake cannot exist as such unless surrounded by land, and a child cannot exist as such unless it has parents, this cannot be regarded as making the species *lake* and *child* self-founding, whereas the species *sibling* clearly *is* self-founding, since a sibling cannot as such exist unless *another* sibling exists. So, using ' $\alpha \sqsupset \beta$ ' for ' α s are founded on β s' we arrive at the following definition of generic foundation:

$$(2) \quad \alpha \sqsupset \beta := \Box (\forall x) (x \in \alpha \supset (\exists y) (y \in \beta \ \& \ x \not\prec y \ \& \ y \not\prec x))$$

where ' $x \not\prec y$ ' abbreviates ' $\sim(x < y)$ '.

The essential nature of the foundation relation is expressed by the prefixed necessity operator. Since we have assumed the availability of the S4 principle $\Box p \supset \Box \Box p$, we have as a consequence that all generic foundation relationships hold of necessity:

$$(3) \quad (\alpha \sqsupset \beta) \supset \Box (\alpha \sqsupset \beta)$$

a result which would meet with Husserl's approval. It might be questioned, however, whether strict implication adequately fits the bill for expressing the relationship between α s and β s that we are aiming for. Should it perhaps involve some relationship of logical relevance, connecting the two species? For instance, would it be better to adopt the following as a definition of foundation:

$$(4) \quad (\forall x) (x \in \alpha \rightarrow (\exists y) (y \in \beta \ \& \ x \not\prec y \ \& \ y \not\prec x))$$

where the arrow represents the *entailment* connective?⁴² This would appear to be in harmony with Husserl's view of the relationship as arising out of the very nature of α s as such. It would preserve the theorem (3) above, since the logical system E of entailment has an S4 modal structure. And suppose that it is necessarily false that there be an α : would

not definition (2) make it trivially true that $\alpha \supset \beta$ for all species β ? This would also appear to favour an approach via entailment. But we have stipulated informally that no such case can arise, because it would violate the requirement that every species be such as to be capable of having instances, so this problem cannot arise as long as we remain within the limits imposed by this stipulation. While it would seem both possible and perhaps in the long run desirable to develop a foundation theory in terms of entailment or some other relevant connective, this course places additional difficulties in the path of interpreting Husserl, and so will not be followed here.

While the definition of foundation given above as (2) includes many important essential part-whole relationships, it does not include them all, so it is worth noting that the wider sense of foundation given by Husserl can be captured as follows:

$$(5) \quad \alpha \supset \beta := \square (\forall x) (x \in \alpha \supset (\exists y) (y \in \beta))$$

It is in this sense, rather than that of (2), that a whole which needs a part of a certain kind may be said to be founded on that part. For instance, it is essential to men that they possess brains, or tables that they possess tops. Such essential parts cannot however be described as being *associated* with the wholes which include them, since associated parts are co-ordinated, neither being the whole itself. It may be because Husserl was not quite clear which of the various possible essentialistic relations he wished to describe as foundation that we get from him more than one, non-equivalent definition. It is more to the point, however, simply to note the differences, remarking that both concepts of foundation have their uses. We shall in what follows concentrate predominantly on ' \supset ', since this appears to carry the greater weight for Husserl. However, as we shall see, some of the results which Husserl takes to hold for foundation in general hold for ' \supset ' but not for ' \supset '.

As an application of our definitions let us consider one of the propositions put forward by Husserl in § 14 of the Investigation, and for which he offers informal proofs. This is Husserl's Theorem I⁴³:

If an α as such requires to be founded on a β , every whole having an α , but not a β , as a part, requires a similar foundation.

To represent this we introduce by definition a complex general term ' $\alpha)\beta$ ', to be read 'object which contains an α but not a β as part'. Defini-

tions of general terms take the form of showing what condition an individual must satisfy to fall under the term, and accordingly have the form ' $t \varepsilon \alpha := (\dots t \dots)$ ', where ' t ' is an arbitrary singular term and ' $(\dots t \dots)$ ' stands for a sentential context containing occurrences of ' t ' but not of ' α ' or any other term defined in terms of ' α '. Thus we give a definition of ' α) β ' as follows:

$$(6) \quad t \varepsilon \alpha) \beta := (\exists x) (x \varepsilon \alpha \ \& \ x < t) \ \& \ \sim (\exists x) (x \varepsilon \beta \ \& \ x < t)$$

It is understood that if an open sentence of the form ' $x \varepsilon \alpha$) β ' occurs in a proof, that when replaced by an open sentence corresponding to the right-hand side of the definition (6), we reletter bound variables if necessary so as to ensure that scope problems do not arise; otherwise the use of open sentences containing defined general terms is the same as that when there are no bound variables present.

Given (2) and (6) it is a simple matter of modal predicate logic, using only the transitivity property of '<' to prove

$$(7) \quad \alpha \sqsupset \beta \supset \alpha) \beta \sqsupset \beta$$

and its necessity, which is the obvious way of representing Husserl's Theorem I.⁴⁴ Thus what Husserl confidently calls its axiomatic self-evidence is seen to stand up in the present formalisation.

While the relation ' \sqsupset ' is trivially reflexive, the relation ' \sqsupset ' is not. Only certain species are self-founding in the stronger sense. The most obvious examples are those using derelativised nouns. These do not figure as such in Husserl's examples, although his exposition uses such nouns a good deal. We can offer the following as examples: sibling, spouse, partner, colleague, cousin, accomplice, companion, fellow, enemy, peer, associate. The last example uses the very idea Husserl employs to characterise the foundational tie as such. We might offer examples of non-self-founding species such as house, mountain, planet.

A crude grammatical test for whether a noun corresponds to a founded species or not is to see whether it is natural to describe an α as, say, an α *of* something or someone. So every colour is the colour of something, every spouse is the spouse of someone, every planet is the planet of some star, every monarch is the monarch of some realm, and so on. The test is only crude, however, in that some founded species are not so spoken of, e.g. we do not call a lake or an island a lake *of* the sur-

rounding land, or island *of* the surrounding sea, and the word 'of' can mean many other things. Nevertheless the test is a useful rough guide. For self-founding species, for instance, it often makes sense to say that every α is an (or the) α *of* another α .

Some foundation relations between species are symmetric; Husserl calls such relations *two-sided* or *mutual* foundation.⁴⁵ For example the species husband and wife, or colour and extension, are mutually founding.⁴⁶ On the other hand, some foundation relations are not symmetric; these are *one-sided*. Thus in Brentano's psychology judgments are one-sidedly founded on presentations or ideas, while feelings of love and hate are founded on judgments, again one-sidedly, and hence indirectly on ideas. To take our geographical example again, a lake is as such one-sidedly founded on dry land. Several terms from physical geography show such one-sided foundation, e.g. mountain, plateau, cwm, island, peninsula, and so on.

Whereas ' \neg ' is transitive, ' \lrcorner ' is not. The definition (2) has to be examined to see why not. The conditions $\alpha \lrcorner \beta$ and $\beta \lrcorner \gamma$ do not suffice to show that $\alpha \lrcorner \gamma$, because if we have $a \in \alpha$, $b \in \beta$ and $c \in \gamma$ satisfying the conditions for (2), the fact that neither a and b nor b and c are part of one another does not suffice to show that a cannot have c as part or vice versa. Examples of this are hard to come by, but the following suggests itself: there cannot be a person conducting the defence at a trial unless there is a trial, and there cannot be a trial unless there is a defendant or defendants. But there is nothing to stop the or a defendant conducting the defence at the trial. Another consideration which reinforces the position that ' \lrcorner ' be not transitive is that if it were, all species two-sidedly founded on some other species, would by transitivity and symmetry be self-founding, in the strong sense, and this is surely not intended.

It is possible to define certain more general concepts relating to foundation if we allow ourselves to quantify over species, introducing bound general term variables. We may say that a species is *founded* or is *founding* according as there is a species it is founded upon or, respectively, founds:

$$(8) \quad \alpha \lrcorner := (\exists \xi)(\alpha \lrcorner \xi)$$

$$(9) \quad \lrcorner \alpha := (\exists \xi)(\xi \lrcorner \alpha)$$

Here we use a notational device which we find convenient elsewhere also: to represent existential generalisation by omission. We can also de-

fine an important concept of essential independence for species: α s as such are *essentially independent* when they are not founded:

$$(10) \quad I(\alpha) := \sim(\alpha \supset)$$

The other important concept of foundation concerns not the relations between species but those between individuals. Husserl, as we mentioned, brushes very lightly over the distinction, and in the 1929 commentary on the formal work in the third investigation by Eugenie Ginsberg, whose later article appears in this volume in translation, the distinction goes quite unnoticed.⁴⁷ If we are to be able to speak of foundational relationships between individuals at all, we cannot rest content with defining such individual foundation in terms of generic foundation, after a fashion such as this: a , as an α , is founded on b as a β := $a \in \alpha \ \& \ b \in \beta \ \& \ \alpha \supset \beta$. The first and most obvious reason is that just because $a \in \alpha$ and $b \in \beta$ and $\alpha \supset \beta$ it does not follow that a is founded on b . For b must be not just any β but *the right one*. If Alice, as a wife, is founded on Bob, as a husband, it is not sufficient that Alice be a wife and Bob a husband: they must be married *to one another*. Similarly, although, to use Husserl's example again, a moment of colouredness requires a moment of extension and vice versa, merely taking the colour-moment of one thing and the extension-moment of another does not yield the more independent colour-extension whole required.

A definition of individual in terms of generic foundation would be forthcoming were we able to specify a condition $F(a,b,\alpha,\beta)$ to be added as a conjunct to the right-hand side of the attempted definition above. I have been unable to find such a general condition, and indeed have come to believe, somewhat reluctantly, that there is none to be found. We can certainly find formulae for many particular cases: for example in the marital case we simply need the relation ' a is married to b '. But it is clear that in this case the species terms are derived from the relative by derelativisation. Furthermore, if Husserl is correct in saying that the foundational relations between species rest on the essential natures of the species in question, and not on formal considerations, we *ought not* to be able to find such a general formula.

One initially promising way of trying to define individual foundation from generic is to introduce different concepts of whole. For the whole formed by the colour of this + the extension of that is a mere whole, in the widest sense mentioned above, whereas the whole formed by the co-

lour of this + the extension of this is much more coherent. This coherence cannot consist in total independence however, since both the colour and extension may (and in this case do) require completion by something beyond them. Nor can it mean simply completability, since the colour of this + the extension of that can also be completed into a self-sufficient whole, namely this + that. Similarly, while Mr. Smith and Mrs. Jones do not form a maritally self-sufficient whole, together with Mrs. Smith and Mr. Jones they do, namely a pair of married couples. This sort of completion results in a whole which is in a certain sense too large, giving us the sum of two of the sort of whole we were looking for. So the whole resulting from completion must be specified more closely. It would seem that the best way to do this is to invoke the concept of a *pregnant* whole. The colour of this + the extension of this is part of a self-sufficient visual datum, say, which is not merely summed aggregatively with another. Similarly a single married couple is the smallest maritally independent whole; every member of the collection is maritally connected to every other, whereas in a pair of couples each member of a couple is maritally unconnected with the members of the other couple. The pregnant whole for the foundation relation in question offers the promise of being neither too large nor too small. But this concept is itself defined in terms of the relation of individual foundation, as we shall see below, so it cannot be invoked without circularity. I do not believe that Husserl saw the threat of circularity here, so it is not to be expected that we could find from his account any indication as to how it might be avoided.

Another suggestion would be that two foundationally related items can only be found together in one substance. This would mean restricting the examples of foundation relations unduly, since a planet would normally be regarded as a substance, yet planets as such cannot exist unless stars exist, for example. The suggestion is quite foreign to the spirit of Husserl's enterprise, for Husserl never speaks of substances. It would I think have been much more to his liking to work towards a definition or definitions of substance through his theory rather than the other way around. This is not to say that we cannot use the notion of substance to guide our investigation in various directions, merely that the general problem of individual foundation is not to be resolved by recourse to the notion. It accordingly seems best that we treat the concept of individual foundation as primitive.

That a particular α cannot exist without a β may be true: that it cannot

exist without the particular β which satisfies this requirement need not also be true: for Bob to be a husband he must be married, but he need not be married to Alice; he might have married Carol instead. So at the level of individual foundation a measure of unavoidable factuality enters in, though not to all relations of individual foundation.

One of the general problems facing a theory of individual foundation is the question as to how far we may be taking Husserl to be working within assumptions about logical form which are implicitly Aristotelian, and whether such assumptions must be rejected. This concerns in particular the question whether an individual may be an instance of different species such that its foundational relations to other individuals vary according to the species in question. For instance, Jupiter falls into the species *planet* and also *heavenly body*. *Qua* planet, Jupiter is foundationally related to the Sun, the substantive relation in this case being gravitational. But *qua* heavenly body, Jupiter is not founded on the Sun. The problem is that if we are to say, as Husserl appears to want to, that a given individual either is or is not founded on another individual, we have to either deny that an individual can belong to two co-ordinate species, or else insist that there is some one privileged species with respect to which all talk about foundational relatedness of an individual is to be carried on. This sort of supposition can be roughly characterised as Aristotelian, and there are indications of such a position in Husserl.⁴⁸ An obvious candidate for such a privileged species is an individual's *infima species*, the product species of all those to which it belongs, which would, on Husserl's view, have only that individual as extension. It would be what he calls an *eidetic singularity*.⁴⁹ The problem with this is that in order for such a species to guarantee individuation of the object in question it would, *pace* Leibniz, have to comprise relational characteristics. This is not in itself objectionable, since many of the clearest cases of foundation rest on relations. But again the contingency of many of the relationships into which a thing enters means that we should have to find a way to distinguish essential from accidental attributes of something in order to arrive at a stable and useful conception of individuals' relative dependence and independence. Also an *infima species* will almost certainly have an infinite intension, so it could not be a working tool for the investigation of individual foundation. To take this problem into account, we shall have to mark explicitly the species under which we are considering an individual's foundedness.

The individual foundational relations hold between individuals not

merely as such, then, but considered as belonging to given species. The only way in which this consideration can be excluded from explicit mention is either by generalising, or by assuming that for certain individuals there are species to which they could not but belong in order for them to exist at all, in other words to assume essentialism for individuals. We shall explore both possibilities.

The basic relation of individual foundation we can accordingly gloss, in full dress, as '*s, qua* α , is founded on *t, qua* β '. We shall symbolise this as ' $s_{\alpha} \bar{\beta} t$ '. The similarity of basic symbol is intentional, but note that it is flanked by singular rather than general terms, and is indexed by a pair of general terms. We must take care to distinguish this formulation from the similar sounding '*s, which is an* α , is founded on *t, which is a* β '. The latter, while mentioning the species to which *s* and *t* belong, does not, like the former, say that it is in virtue of belonging to these species that they are so related. If we take '*s* is founded on *t*' as merely meaning that *s* and *t* belong to *some* species whereby they are so related (cf. (20) below) then the latter form may be true while the former is false: e. g., it is true that Jupiter is a heavenly body, and is founded on the Sun, which is also a heavenly body, but it is not in virtue of Jupiter's being a heavenly body that it is founded on the Sun, but rather in virtue of its being a planet of the Sun.

Expressions like 'as such', '*qua*', 'in virtue of being', and others repeatedly used by Husserl and by ourselves in discussing foundation, are logically peculiar in that they do not form unrestrictive relative clauses as 'which', 'that' etc. do, but create an intensional context. To see this, let us take a pair of examples. Suppose the owner of the Casa Negra night-club is also the husband of Dolores, its principal singer. Then while the following are true:

- (a) The owner of the Casa Negra cannot exist as such unless the Casa Negra exists.
- (b) The husband of Dolores cannot exist as such unless Dolores exists.

The sentences obtained by interchanging subjects of (a) and (b) are false. Similarly, supposing that all and only rational animals are featherless bipeds, it does not follow that

- (c) A rational animal as such (by nature) has two legs.
or that
- (d) Jones, *qua* rational animal, has two legs.

It follows that there is no such entity as Jones *qua* rational animal, which is to be distinguished from Jones *qua* loving father, for instance. Expressions like 'Jones *qua* loving father' are not genuine singular terms, but sentential fragments having the force e. g. of 'Jones is a loving father, and as such, he . . .' where the 'as such' creates the intensional context.⁵⁰ This property of 'as such' and related expressions throws into relief the difficulties about the connection between generic and individual foundation. To the extent that we have either to mention or otherwise assume, with expressions like '*qua*' or 'as such', a general kind or species, Husserl is right in taking individuals to stand in foundational relations in virtue of their belonging to species which stand in generic foundational relations.⁵¹

We can give a specification of the connection between generic and individual foundation by the following axioms:

- (11) $\Box (s \overset{\neg}{\alpha} \beta t \supset (s \varepsilon \alpha \ \& \ t \varepsilon \beta \ \& \ \alpha \neg \beta \ \& \ s \prec t \ \& \ t \prec s))$
 (12) $\Box (\alpha \neg \beta \supset (\forall x) (x \varepsilon \alpha \supset (\exists y) (x \overset{\neg}{\alpha} \beta y)))$

The converse implication to that given in (12) follows from (11) together with the definition (2) of generic foundation. This gives us the desirable result that α s as such are founded on β s if and only if any α is as such founded on some β as such. The appearance of triviality of this result disappears when it is remarked that 'founded on' does not mean the same in both occurrences. The indefinability of the individual relation in terms of the generic amounts to the lack of a general formula $F(s, t, \alpha, \beta)$ which could be added as a conjunct on the right of the implication in (11) so as to turn it into an equivalence.

§ 4 Dependence

Having dealt at length with the problems of foundation, we should now turn to the more general concepts of dependence and independence, which will of course vary according to the conception of foundation by means of which they are defined. Given the definition of foundedness (8) and essential independence (10) for species, we can define related notions of dependence and independence for individuals: an individual is *partly dependent*, written 'dep', when some species it belongs to is founded:

$$(13) \quad \text{dep}(s) := (\exists \xi)(s \varepsilon \xi \ \& \ \xi \neg)$$

while an individual is *totally independent*, written 'ind', when it is not partly dependent, that is:

$$(14) \quad \text{ind}(s) := (\forall \xi)(s \varepsilon \xi \supset I(\xi))$$

One could similarly define partial independence and total dependence. It follows from (11), (12) and (14) that an individual is totally independent if and only if it is not founded in any way on any other individual. It should be noted that by definition no individual can be self-founding, since every individual is a part (albeit improper) of itself. By contrast, some species are, as we have seen, self-founding.

The condition of total independence is extraordinarily strong, because of the universal quantification. It might be wondered what, if anything, could satisfy it. Since, according to orthodox cosmology, God falls under the term 'creator', and there can be no creator without creatures, even God would not, according to this view, be totally independent, being reciprocally founded on his works.

Because of the strength and uncertainty of application of such conditions, it would appear advantageous to develop more readily applicable conditions. One way to do this is to attempt to distinguish in individuals those species to which they belong of necessity from those to which they belong adventitiously. It is here that we shall use Wiggins' *de re* operator 'nec'. This will be used to offer a faithful formal rendering of such expressions as '*s* must be an α ', '*s* is essentially/necessarily/by its very nature an α '. This would normally be written, using property-abstraction, as

$$[\text{nec } (\lambda x)(x \varepsilon \alpha)](s);$$

however, to avoid unnecessary symbolic complication, I shall adopt the abbreviation

$$s! \varepsilon \alpha$$

and in general, for any *simple* predicate, where there is no risk of confusion, *de re* necessity will be marked by an exclamation mark after the occurrences of terms of which the predicate holds of necessity; so '*s*! < *t*!' will be short for $[\text{nec } (\lambda x)(\lambda y)(x < y)](s, t)$ and so on.

We may now define an individual as being *essentially independent*, written ‘essind’, when every species to which it belongs of necessity is independent:

$$(15) \quad \text{essind}(s) := (\forall \xi)(s! \varepsilon \xi \supset I(\xi))$$

while an individual is *essentially dependent* when it is not essentially independent:

$$(16) \quad \text{essdep}(s) := (\exists \xi)(s! \varepsilon \xi \ \& \ \xi \neg I)$$

Armed with these new concepts we may resolve the theological problem about God’s dependence on the world, in a manner suggested by Aquinas,⁵² by noting that since God need not have created the world, his being a creator is not essential to him, so he can be secured essential independence. The world, on the other hand, is essentially dependent, at least according to the traditional cosmology. The only possible candidate for total independence on such a view would be the totality comprising both God and the world.

According to traditional theology, the world is dependent on God both because he created it and because he continuously sustains it. The Husserlian concept of dependence covers both kinds of dependence, because it makes no reference to time. So something which needs to be produced by something else, but which can thereafter survive without this, is dependent on it in a different way from that in which something is dependent on something which it requires to exist at every time at which it exists itself. It is worthwhile contrasting the views of Husserl on dependence with those of his student Ingarden. In his chief work, *Der Streit um die Existenz der Welt*, Ingarden distinguishes four basic senses of dependence/independence.⁵³ Since these are given in opposing pairs, we need only characterise one of each pair. They may be set out in a table as follows:

- | | |
|----------------------|-----------------------------|
| (1) Autonomy | – Heteronomy |
| (2) Originality | – Derivation |
| (3) Self-sufficiency | – Non-self-sufficiency |
| (4) Independence | – Dependence. ⁵⁴ |

An object is autonomous or self-existent if it has its existential foundation in itself, is immanently determined.⁵⁵ An object is original if, in its

essence, it cannot be produced by any other object.⁵⁶ An object is self-sufficient if it does not need, by virtue of its essence, to coexist with something else within a single whole.⁵⁷ Finally, an object which is self-sufficient is independent if it does not require, by virtue of its essence, the existence of any other object which is also self-sufficient.⁵⁸ Ingarden draws attention to Husserl's examination of dependence and independence in the third investigation, but regards the eight concepts he sets out as belonging to a kind of theory which Husserl did not recognise, which Ingarden calls existential ontology, and which he contrasts with both formal and material ontology. There are considerable differences of background between Husserl's and Ingarden's respective treatments of dependence and independence, which we cannot enter into here. It is clear however that Ingarden's distinctions (2)–(4) could be variously interpreted within Husserl's theory of foundation. Ingarden in particular models his (3) on Husserl's definition of foundation. The difference between (3) and (4) is not highlighted by Husserl, and in making it Ingarden must have in mind some concept of whole stronger than the wide concept employed by Husserl. The only one of Ingarden's pairs which does not obviously fall within the general Husserlian account of foundation is (1).⁵⁹

If an object t is essentially independent, it follows that it is possible that t has no supplement, i. e. that t could constitute all there is, there being no whole (in the wide sense) of which t were a part. This possibility, which shows the self-sufficiency of the object in a perspicuous light, coincides with the conception of an object which is something for itself (*Etwas-für-sich*) in the late ontology of Brentano, as formulated by Chisholm:⁶⁰

$$(17) \quad t \text{ is Etwas-für-sich} := \Diamond \sim (\exists x)(t \ll x)$$

This coincidence of notions is an interesting sidelight on the otherwise very different worlds of Husserl's ontology of the *Logical Investigations* and the ontology of Brentano in the *Kategorienlehre*. It suggests that Husserl's concepts of dependence and independence could contribute valuable insights to the problem of substance, which looms much larger for Brentano.

If $s \sqsupset_{\alpha} t$ then, as we have pointed out, it may be quite accidental to t as such, even t as a β , that it should *satisfy* the requirement for s for a β . One fact which shows this clearly is the possibility in certain cases of *dis-*

junctive satisfaction. Suppose, for instance, that Brown is a cat-owner. Then, as such, he must possess some cat. But he may possess more than one, each of which would, on its own, be sufficient to render him a cat-owner. At any time at which he owned more than one cat, the loss of one would not affect his status as a cat-owner. Indeed, provided he replaced cats as they died or he lost them etc., he could, barring catastrophe, remain a cat-owner for a time-span far longer than the life of any of his cats. In a similar way, a man is biologically dependent for continued life upon a regular supply of oxygen, water and nutrients, but the particular consignment of such material which actually sustains him will vary widely over time.⁶¹ Similar considerations apply to those parts of a thing which are essential to its being the sort of thing it is, but which can suffer replacement without the thing's ceasing to exist, either because it has more than one, and can acquire more as need be, or if it can temporarily survive without one. The replacement of cells in organisms gives an example of the first kind, while the repairing of machines gives one of the second.⁶²

Having defined the dependence and independence, whether essential or not, of individuals, we should now define *relative* dependence and independence, concepts of which Husserl makes much use in the investigation. We first define some more general concepts of individual foundation, following the practice established earlier of marking generalisation by omission of symbols.

$$(18) \quad s_{\alpha} \sqsupset t := (\exists \xi)(s_{\alpha} \sqsupset_{\xi} t)$$

$$(19) \quad s \sqsupset_{\beta} t := (\exists \xi)(s_{\xi} \sqsupset_{\beta} t)$$

$$(20) \quad s \sqsupset t := (\exists \xi \eta)(s_{\xi} \sqsupset_{\eta} t)$$

The general concept of foundation given by (20) does not make it explicit why s is founded on t . This more general concept frequently occurs in Husserl's exposition.

Husserl defines relative dependence as follows:⁶³

A content α is relatively dependent with regard to a content β (or in regard to the total range of contents determined by β and all its parts), if a pure law, rooted in the peculiar character of the kinds of content in question, ensures that a content of the pure genus α has an *a priori* incapacity to exist except in, or as associated with, other contents from the total ranges of the pure genera of contents determined by β .

I have quoted this in full because it illustrates vividly the sorts of problem of interpretation we face in the investigation. In the middle of what purports to be a definition, which should therefore be totally unambiguous, one finds inserted hedges and adjustments, which make a significant difference to the sense. It also displays Husserl's indifference to the possible problems of an individual's belonging to various species, since the same schematic letters are used for species and for members of these.

Three possible concepts of relative dependence suggest themselves to me on the basis of this passage. The first is that relative dependence is nothing other than individual foundation. This is arrived at by simply ignoring the bracketed adjustments in the passage and the clause 'or as associated with'. This identification may not be exact, because of the ambiguity of the phrase 'determined by β ', which might refer to parts of β , or essential parts of β , or simply some species to which β belongs (it must be remembered that here we are following Husserl's ambiguous lettering). It may be that the concept of individual dependence here suggested is not quite the same as that given by our (11)–(12).

By taking account of the adjustments beginning 'or . . .' we may arrive at the reading that s , say, is not directly founded on t but on something 'in its range', i. e. something which is, in the widest sense, a part of t . So we have the following alternative concept of individual relative dependence:

$$(21) \quad \text{dep}_1(s,t) := (\exists x)(x < t \ \& \ s \sqsupset x)$$

According to (21) anything which is founded on something else is thereby dependent₁ with respect to it, a result which is quite in the spirit of Husserl's exposition. The converse to this is not true: an object may be dependent₁ on another without being founded on it. To take an example from Eugenie Ginsberg's discussion of the Investigation, the shape of a particular brick is founded upon other aspects of the brick, and so this individual shape is dependent₁ upon the wall of which the brick happens to be a part, yet the shape could hardly be said to be founded upon the wall.⁶⁴ Ginsberg does not however distinguish between foundation and relative dependence, and so some of her attempts to show that Husserl's theorems are not all valid are vitiated. The wide concept of dependence₁ here canvassed is perhaps somewhat unnatural, and we should perhaps take closer cognisance of the phrase 'the total range of contents determined by β and all its parts'. There is, I think, no telling exactly

what this phrase is intended to mean, but the Ginsberg example suggests that we choose not merely an adventitious part of the whole t as something upon which s is founded, but rather take a part which t could not but have, i. e. something u such that $u < t!$, using our abbreviated device for showing essential predicates. This then suggests a third possible concept of relative dependence:

$$(22) \quad \text{dep}_2(s,t) := (\exists x)(x < t! \ \& \ s \sqsupset x)$$

According to this sense, whenever an individual is founded on another, it is also dependent₂ upon it, since for any individual t it is true that $t < t!$.

Dependence₂ does not reduce to foundation however. For one thing, s may be dependent₂ on t and at the same time a part of it, which means that, according to (11), s cannot be founded on t . In a case such as this we may say that, in one sense at least, s is a *dependent part* of t :

$$(23) \quad \text{deppt}_2(s,t) := \text{dep}_2(s,t) \ \& \ s < t$$

while of course it is similarly possible to define another sense of 'dependent part' through dep_1 :

$$(24) \quad \text{deppt}_1(s,t) := \text{dep}_1(s,t) \ \& \ s < t$$

It is clear of course that if $\text{deppt}_2(s,t)$ then $\text{deppt}_1(s,t)$: the second sense is stronger than the first. The close connection between foundation and dependent parts may be seen by the following theorem:

$$(25) \quad s \sqsupset t \supset \text{deppt}_2(s,s+t)$$

where $s+t$ is the aggregate or sum of s and t : the theorem follows from definition (23) together with the result that $t < (s+t)!$; clearly the very sum $s+t$ could not but have had t as part.⁶⁵ This shows that anything which is founded on something else is thereby a dependent part of a whole which is more comprehensive than either the founded or the founding part. It is because of this that the three notions of foundation, dependence, and being a dependent part, are so readily confused. It may be that Husserl himself did not make the distinctions so clearly as we have drawn them, but there is, as has been shown, sufficient evidence

from the investigation to show that such fineness of distinction can, and was perhaps intended to be, read from the text.

Recalling the supposition that entities may essentially belong to certain species, being the individuals they are, we can introduce various notions of essential foundation and dependence which are stronger than those we have used hitherto. We can for instance describe an individual s as *essentially founded* on an individual t when s is founded on t in virtue of some species to which s belongs essentially:

$$(26) \quad \text{essfd}(s,t) := (\exists \xi)(s! \varepsilon \xi \ \& \ s \sqsupset_{\xi} t)$$

while t essentially founds s when s is founded on t through a species to which t must belong:

$$(27) \quad \text{essfg}(s,t) := (\exists \xi)(t! \varepsilon \xi \ \& \ s \sqsupset_{\xi} t)$$

and a yet stronger relation can be obtained either by conjoining these, or, stronger yet, by insisting that the species α, β such that $s \sqsupset_{\alpha} t$ are such that $s! \varepsilon \alpha$ and $t! \varepsilon \beta$.

We have already mentioned the possibility of disjunctive or generic satisfaction of an individual's need by other individuals. For though s may be in some sense essentially founded on t , this may not mean that s could not have been essentially founded on something other than t satisfying the same requirement. To take a biological example, an organism as such is, let us suppose, essentially founded at any time on some consignment of water, but any other consignment would have done equally well. Similarly an internal combustion engine is essentially founded on a supply of lubricant (here it is obvious that we mean a functioning engine, not a museum-piece), but again which particular mass of lubricant does the job is not important. A ship-launching ceremony might be thought to be essentially founded upon a bottle of champagne, but it need not have been just the one which was used. In other cases, however, an individual s is not only essentially founded on some other thing t , but it could only have been t upon which it was so founded. In such cases we may introduce definitions based on formulas such as $s \sqsupset t!$, $s! \varepsilon \alpha \ \& \ s \sqsupset_{\alpha} t!$, and $t! \varepsilon \beta \ \& \ s \sqsupset_{\beta} t!$; which can themselves be used to further define notions of dependent parts. So we might be then equipped to say in what sense it is essential to a man that he has not just any brain, but this very brain, whereas it is not essential to him that he have this very

heart, or in what sense it is essential to a person that he or she should have the very parents he or she did have.⁶⁶ There is perhaps at present little point in doing more than indicating that there is here a wide range of questions and issues, some of them bearing on regularly-debated issues such as personal identity, together with a rich fund of possible concepts of dependence, all developed out of Husserl's ideas, and requiring further refinement.

We can however indicate a possible formulation of Husserl's attempt to define the pregnant concept of whole in terms of foundation. We need here individual foundation, as was argued earlier. Firstly we define *direct foundational relatedness*: two things are directly foundationally related when one is founded on the other:

$$(28) \quad \text{dfr}(s,t) := s \supset t \vee t \supset s$$

Then we define *foundational relatedness* as the (proper) ancestral of the relation of direct foundational relatedness:⁶⁷

$$(29) \quad \text{fr}(s,t) := \text{dfr}^{>0}(s,t)$$

Thus two entities are foundationally related if one founds the other, or both found or are founded on some third thing, or one founds and the other is founded on some third thing, etc. Then an entity is a *pregnant whole* when all its parts, in this case its proper parts, are foundationally related to one another, and no part is foundationally related to anything else outside this entity:

$$(30) \quad \text{Prwh}(s) := (\forall xy)(x \ll s \supset ((y \ll s \ \& \ x \neq y) \equiv \text{fr}(x,y)))$$

While it is thus not too difficult to express Husserl's idea symbolically it is much harder to see what it amounts to in practice. In theory the world should partition itself neatly into discrete entities, each of which is a pregnant whole. (Entities are discrete when they have no common part.) It might however be the case that every entity is foundationally related to every other, in which case there would be no partition, and only one pregnant whole, the world itself. This result would certainly be counted as in some sense monistic. It is possible that the sort of whole which Husserl had in mind when discussing pregnant wholes would be lesser in extent; to capture such wholes we might need to take a tighter foundation

relation as the basis for the definition of foundational connectedness. In general, the stronger such a relation, the tighter the organisation of the resulting wholes, the smaller in extent they are, and the more there are of them. So it seems that rather than there being a single concept of pregnant whole, there are several, having in common a recipe for generation from a concept of individual foundation. This is a characteristic outcome of studying the third investigation: ideas which at first sight seem sharp show themselves to hide various possible interpretations.

§ 5 Husserl's Six Theorems

An illustration of the difficulty is the attempt to interpret the six theorems of § 14: one has to use these as a guide to what Husserl meant at the same time as attempting to see whether they are valid or not. It is instructive to examine these and Husserl's proofs for them. We already saw above how Theorem I, interpreted as (7), is valid. Here is Theorem II:⁶⁸

A whole which includes a non-independent moment without including, as its part the supplement which that moment demands, is likewise non-independent, and is so relatively to every superordinate independent whole in which that non-independent moment is contained.

Husserl states that this follows from Theorem I as a corollary, given a definition of relative dependence. But he is wrong in this. Theorem I is stated in terms of species, whereas Theorem II relates to individuals. Here is a place where the transition between these two levels is not so simple as Husserl believes. We can give an example of things satisfying the intuitions represented by Theorem II which do not in any obvious way satisfy those of Theorem I. Let us call any expression which requires completion by only names or other singular terms to yield a sentence a *predicate*. Then the English verb 'loves' is a predicate, requiring completion by two names to obtain a sentence. In the sentence 'John loves Mary' the names 'John' and 'Mary' satisfy this double requirement. Now the predicate 'loves Mary' also has a requirement for supplementation by a name, and in the given sentence this requirement is met by the name 'John'. We might say that in the given sentence the predicate 'loves Mary' *inherits* from the predicate 'loves' that requirement which is met by the name

'John'.⁶⁹ This is in conformity with the way in which the first part of Theorem II is phrased: the predicate 'loves Mary' does not contain *all* the supplements demanded by its part 'loves', and so inherits from the latter the demand satisfied by 'John'. But the most obvious way of expressing this in the terms of Theorem I is to substitute the term 'predicate' for ' α ' and 'name' for ' β '. But in that case we should render ' $(\alpha)\beta$ ' as 'predicate which does not contain a name as part': but precisely 'loves John' *is* a predicate which contains a name as part. It may be that this particular kind of multiple satisfaction was not considered by Husserl in his phrasing of Theorem I. To show that Theorem II does indeed follow from Theorem I we should have to be assured that *whenever* we have things satisfying the premisses in Theorem II we can always find a pair of species α and β such that Theorem I is satisfied with respect to the supplement which the larger whole inherits from the smaller moment. It seems to me dubious that we should be able to establish this in full generality, so it may be that Husserl's theorems require another axiom to support them, such as the following:

$$(31) \quad (s \supset_{\beta} u \ \& \ s < t \ \& \ u \not< t) \supset t \supset_{\beta} u$$

Some such principle does indeed seem to be taken as self-evident by Husserl, but it cannot be directly proved from the proof of Theorem I, because it is compatible with the principles of this theorem that a is an α , b is a β such that $a \supset_{\beta} b$, and that c is an $(\alpha)\beta$, and so itself requires a β for completion, but rather than inheriting a 's requirement satisfied by b , its requirement is satisfied by some further β , say b' . It is hard to find a convincing example of this state of affairs, which leads me to concur with Husserl. The nearest to a counterexample that I have managed is this: let ' α ' be replaced by 'represented district' and ' β ' by 'representative': the relevant whole being a district together with its representative. Now a council ward may be part of a parliamentary constituency, but the constituency, even if it does not contain the councillor who represents the ward, does not inherit the requirement for him, but has its own requirement met by its Member of Parliament. However, the force of this purported counterexample is somewhat blunted by the possible ambiguity in the notion of 'district', which might, one may say, have a bare geographical meaning and a more sophisticated administrative one. It might be argued that it is only in the administrative sense that a district's representation requirements arise, whereas it is only in the geographical

sense that the ward is part of the constituency. In administrative terms the ward is not part of the constituency, but a completely different entity entering into quite different governmental arrangements. It is here that we face the problem of whether it is one and the same thing which is both a council ward and part of the parliamentary constituency, or rather whether these two *coincide*.

Given such uncertainties, it is far from apparent that Theorem II is, as Husserl takes it to be, a mere corollary of Theorem I. For this reason I shall confine myself to discussing the *consequences* of (31) taken as axiomatic, together with our other assumptions, rather than attempt to establish (31) or something like it. It can be seen that Theorem II follows very readily from (31), in its two parts, if interpreted as follows:

- (32) $(s \cap u \ \& \ s < t \ \& \ u \not< t) \supset \text{dep}_1(t, u)$
 (33) $(s \cap u \ \& \ s < t \ \& \ u \not< t \ \& \ u < v) \supset \text{dep}_1(t, v)$

In fact we can show not just (32), but the stronger formula obtained by replacing the consequent of (32) by ' $t \cap u$ '. Further, there does not appear to be any need for Husserl to restrict the superordinate wholes v merely to those which are independent. With these minor reservations, we can endorse Husserl's Theorem II provided we are prepared (a) to gloss 'dependent', as 'dependent₁' and provided (b) we accept (31).

Husserl's Theorem III is given in two versions: these both in effect amount to the transitivity of the relation 'is an independent part of'. We shall use therefore a simple version:

If s is an independent part of t and t is an independent part of u then s is an independent part of u .

To clarify this we must first give a definition of 'independent part'. The obvious one will do:

- (34) $\text{indpt}_1(s, t) := s < t \ \& \ \sim \text{dep}_1(s, t)$

One could also define similarly a relation indpt_2 based on the relation dep_2 but the one we have given here fits the bill more closely. For in the presence of (31–3) it becomes easy to prove that

- (35) $(\text{indpt}_1(s, t) \ \& \ \text{indpt}_1(t, u)) \supset \text{indpt}_1(s, u)$

by much the method Husserl uses in his informal proof of Theorem III, except that Husserl appeals both to Theorems I and II, whereas, because of the difficulties we have alluded to, we appeal only to (31) and its consequences.

Irrespective of the merits of (31), Husserl's fourth theorem is valid. His formulation is:⁷⁰

If s is a dependent part of a whole t , it is also a dependent part of every other whole of which t is a part.

We can represent this as

$$(36) \quad (\text{deppt}_1(s,t) \ \& \ t < u) \supset \text{deppt}_1(s,u)$$

and it follows immediately from the definition of deppt_1 and the transitivity of the part-whole relation ' $<$ '. In fact it is a more general thesis that

$$(37) \quad (\text{dep}_1(s,t) \ \& \ t < u) \supset \text{dep}_1(s,u)$$

It should be noticed that this was the assumption questioned by Ginsberg in her brick example, and the principle is harmless once the difference between individual foundation and the more general relation of relative dependence, in the sense of dep_1 , is made clear. One particular restriction of (36) yields the transitivity of deppt_1 . It must be noted that both deppt_1 and indpt_1 are transitive, but that the former is in many ways the more obvious notion. For as Husserl defines relative independence, it does not entail independence *tout court*, whereas this is true for relative dependence. The reason can be seen in the notion of independent part. That a is an independent part of b means only that a is not founded on anything within the range of b ; it does not mean that there is not something else outside b upon which a is founded. Husserl states this explicitly as his Theorem V: to represent this we must give some derelativised notions of dependence and independence derived from the relative notions we have been using. It is for instance possible to define ' s is founded' as meaning simply ' s is founded on something', and similarly for ' s is dependent'. But because of the interrelation between dep_1 and \neg these amount to the same thing, so we shall simply say

$$(38) \quad \text{dep}_1(s) := (\exists x)(\text{dep}_1(s,x))$$

and define something as independent₁ when it is not dependent₁:

$$(39) \quad \text{ind}_1(s) := \sim \text{dep}_1(s)$$

The nice thing about this definition is that we can link now the notion of independence and dependence of an individual previously given as (13–14) in terms of its membership of a species, with the new derelativised notions stated in terms of individuals; by virtue of the principles (11–12) the following is a theorem:

$$(40) \quad \square (\forall x)(\text{ind}(x) \equiv \text{ind}_1(x))$$

so naturally the two contraries, dep and dep_1 , are necessarily equivalent also. This shows that the detour through relative dependence and independence brings us back to the same position as we started from when considering the generic concept of foundation.

Husserl's Theorem V simply says

A relatively dependent object is also absolutely dependent, whereas a relatively independent object may be dependent in an absolute sense.

and we can see how, in our interpretation, this is unproblematically correct.

The final Theorem VI reads

If a and b are independent parts of some whole c , they are also independent relative to one another.

If we render this as

$$(41) \quad (\text{indpt}_1(a,c) \ \& \ \text{indpt}_1(b,c)) \supset \sim (\text{dep}_1(a,b) \vee \text{dep}_1(b,a))$$

then brief consideration shows that it is true, for were either a or b dependent on the other, since each is a part of c , the dependent one would by definition be dependent on c , contrary to the assumption; this is precisely the form of reasoning followed by Husserl in his proof.

We can thus see a way through the six theorems of § 14. Given the axioms and definitions hitherto suggested, the principle (31), which Husserl took to be self-evident, and the selection of dep_1 and not dep_2 as the relevant notion of dependence, all six follow. It is suggested then that

this constitutes an acceptable interpretation of what Husserl meant, which has the merit of making the theorems all valid if the axioms (11–12,31) are valid. This verdict on the semi-formal work of § 14 may be contrasted with that of Ginsberg,⁷¹ whom we suggested did not separate individual foundation from relative dependence, and whose criticisms of Husserl cannot therefore be accepted.

If, as suggested earlier, there are various possible concepts of dependence and independence which we could formulate without being unfaithful to Husserl's intentions, then it would be necessary to test these against the six theorems of § 14 in much the same way as we have done for the concepts connected with dep_1 . But the tests would be more complex, because of the essentialistic nature of many of the stronger definitions. After § 14 Husserl moves on to discuss various other whole-part notions which can be defined in his terms, such as mediate and immediate parts, abstractum and concretum, etc. These will obviously inherit any ambiguities possessed by the basic notions. Rather than follow up all the various possible interpretations, I shall instead turn to possible applications of Husserl's concepts within ontology. Applications in grammar, in particular the question of the dependence-status of different sentence-parts, and the structure of sentences, I hope to deal with elsewhere. For a summary of other applications which have been made, the reader should consult the essay by Smith and Mulligan earlier in this volume.

§ 6 Applications

One problem which was very much a live issue in Husserl's day, but which subsequently became buried, is the question of a distinction between ordinary or genuine objects and objects of higher order.⁷² Such a distinction was fundamental to Meinong's theory of objects, and suggests a kind of logical or ontological atomism whereby the basic objects are those of lowest order, there being aggregates, classes and complexes constituted on the basis of these. Husserl's account of categorial objects, or objects of the understanding, is very much in the same vein,⁷³ and Ingarden too, defends the difference between his concepts of self-sufficiency and independence by invoking this distinction.⁷⁴ Findlay has suggested, in commentary on Meinong, that the implied atomism is untenable.⁷⁵ We have, in Husserl's concepts of the third investigation, the

wherewithal for re-examining the issues. It may be simply misleading to regard objects with other objects as their pieces as somehow less self-sufficient than the pieces. The organs of an organism, while pieces of the organism in the sense that they are both separately presentable and physically separable,⁷⁶ considered as living tissue they are dependent for their continued existence on that of the organism of which they are part; in this sense they are moments rather than pieces of it. For the most vital organs, this dependence is reciprocal. The way is quite open to allow that some larger objects are in fact more self-sufficient than their smaller parts. One example which is mentioned by Husserl,⁷⁷ and which Findlay also cites as militating against the atomistic view of objects,⁷⁸ involves time. Temporal durations, considered not merely as abstractly extended parts of an abstract extended whole, but as concretely occupied by events and processes in the natural world, can no longer be seen as mere pieces, but must be regarded as dependent parts or moments of the whole. This suggests that the ontology which conceives of the world as made up of four-dimensional entities, of which the familiar three-dimensional objects of everyday experience constitute merely temporal cross-sections, is mistaken in supposing that temporally determined objects are sliceable in time in just the same way as a thing is sliceable in space. The theory of four-dimensional space-time objects can be accused of failing to distinguish between things and processes.

A similar consideration might help to dampen somewhat that perennially appealing aspect of all forms of atomism, micro-reductionism. If an entity can be shown to be complex, to consist of parts in a determinate relation to one another, it is the assumption of micro-reductionism that everything which could be meaningfully said about the complex could be expressed mentioning only its parts and their properties and relations. There is no doubt that in many areas of empirical investigation our understanding of entities is furthered by seeing how they are put together. The gains in understanding achieved fuel the drive to find ever more fundamental particles or constituents of matter in physics. It is sometimes suggested that there is no end to how far such reductions can be carried. But the assumption need not go unchallenged. At some stage of our knowledge of the physical world it might be reasonable for the philosopher to suggest that the bunch-of-grapes model of complexity is not the appropriate one. This might occur when the known fundamental particles fall into families by their characteristics, but there has been a prolonged inability to isolate the supposed constituents of these. Rather

than seeing the particles as consisting of more fundamental ones held together by a particularly strong natural glue, it might be hypothesised that the more fundamental parts of the isolable particles are not pieces but moments, which are mutually founding. As Husserl pointed out, such parts need not have any other part or constituent whose job was to hold them together, but require each other by their very nature. Such moments might be compared with the distinctive features of phonological theory,⁷⁹ which cannot be isolated but which explain the resemblances of phonemes, which can.

A rather similar but less universally appealing kind of unifying reduction of explanation is reduction *upwards*, macro-reduction, which seeks explanation of phenomena in terms of the objects in question belonging to some more inclusive totality with its own properties, a whole of which they can be seen to be mere moments. The supreme macro-reductionist was Hegel. Like the micro-reductionist, the macro-reductionist claims that nothing gets lost in his reductive explanation. An intermediate position might contend that micro- and macro-reductionism make opposite but cognate mistakes, the micro-reductionist taking all part-whole relations as relations of piece to whole, while the macro-reductionist takes all such relations as relations of moment to whole. The benefit of the observations drawn from Husserl is not just that it gives us a way to draw the parallels between the atomist and the holist, but that because there are various possible senses of dependence and cognate concepts, it can be made clear that there is not just one possible atomism or holism, but several, so that atomism of one kind might be quite compatible with holism of another. The atomist who sees a man as an aggregate of particles, and the holist who sees him as a mere mode or moment of some greater whole, may simply have different criteria for what it is to be an independent whole.

The question as to what constitutes a *natural* whole is probably not one which could receive a single answer. Which entities constitute natural wholes is something which cannot be settled *a priori*, but must be the concern of the empirical sciences. The sorts of object which we consider as having a tightness of organisation making it fitting to call them wholes in a natural sense seem to have a greater degree of causal coherence, and relative causal isolation from outside phenomena, than those which we should be less inclined to describe as natural wholes. The necessity to speak in terms of *degrees* of isolation and coherence suggests that there can be a spectrum of natural wholes of which some are more clearly

units than others. The paradigmatic examples of natural wholes would appear to be organisms, although these too can be from certain points of view taken as mere moments of some greater whole, involving say a species or an eco-system, while from other points of view they are aggregates of other wholes, such as cells, molecules etc., which have an integrity of their own. Other natural unities are not dissimilar from organisms e. g. in the manner in which they are able to utilise energy. Thunderstorms and river-systems have been suggested as examples.⁸⁰ Aristotle considered stars were not only natural but living unities, an opinion which is by no means so implausible as it appears at first sight.⁸¹ Such a readiness to see analogies between living or organic unities and other natural wholes need be neither anthropomorphic nor need it deny the ubiquity of causal explanation, since it is precisely the causal integrity of a natural whole or system which binds it together. This is not something imposed on reality from outside by our mode of cognition, but represents organisation which is intrinsic and which we discover.

According to this way of considering the multiplicity of ways in which things are connected in the physical world, the distinction between lower- and higher-order objects need not be an absolute one, with a single bedrock layer of natural units, but an object may be from one point of view a natural unit, from another it may coincide with an aggregate of differently organised units, or again be a moment of a greater whole. The fact that objects are naturally organised in many ways ensures that this relativity is not the mere imposition of a conceptual scheme on an otherwise unstructured world, but cuts along natural seams in reality.

When we move from considerations of units in nature to units in other spheres, such as social, legal and economic wholes, causal considerations are no longer so predominant, although they still apply. The unity of many man-machine wholes, such as a manned vehicle, is still predominantly one of relative causal self-containedness, while that of social wholes such as clubs, families, societies, or the various differently-sized units in an army or a business enterprise, require further considerations relating also, e. g., to functions and lines of control or authority. Such considerations may cut across those of causal or spatio-temporal proximity. It is, again, the merit of the vocabulary developed by Husserl that such matters can be discussed without an undue reliance on metaphor, and in full recognition that there will be very many different kinds of relation constituting the various kinds of whole brought into consideration.

One strand in the skein going to make up the traditional notion of substance is that a substance is what exists by itself, without needing the existence of anything beyond itself. In Husserl's terms, such an object is absolutely independent. Given the many different possible senses of 'independent' we could envisage various different senses of 'substance'. It might indeed be the case that some of the historic disputes over substance could be clarified by showing how different philosophers were operating with different concepts of independence. It is noteworthy that Husserl nowhere speaks in the third investigation of substance. His account is furthermore purely formal, and proceeds without assumptions as to which sorts of object are the most basic or paradigmatic independent wholes.

§ 7 Relations and Foundation

We have mentioned in several places the importance of relations between parts of a whole in constituting it as the whole it is. Many of our examples used nouns with a clearly derelativised sense, such as 'husband', 'sibling' and so on. We can very often generate one or more such nouns from a relative term, sometimes artificially. Sometimes the derelativised nouns are common enough to be etymologically unconnected with the relative term in question, as e. g. 'husband' and 'wife' have no etymological connection with the relative 'is married to', and may indeed be far more familiar than the relative notion which defines them. The term 'lake' for instance corresponds to no cognate verb expressing the relation of being land surrounding an expanse of water. Generally speaking, the more closely related things are affected in their properties by their particular relation, the more likely we are to have derelativised nouns to describe the relata as such. This is a partial explanation for the richness of the vocabulary of derelativised nouns dealing with human social and kinship relations, for the relations human beings have to one another mark and are marked by characteristic forms of behaviour of the people concerned.

It might be thought that we always can generate a foundation relation whenever we can obtain a pair of derelativised nouns from a relative term. Suppose for instance that given any binary relation R we define a pair of nouns by derelativisation as follows:

- (42) $t \in R_1 := (\exists x)(tRx)$
 (43) $t \in R_2 := (\exists x)(xRt)$

Does it follow automatically that $R_1 \supset R_2$ and $R_2 \supset R_1$? The answer is no: while we automatically get that $R_1 \supset R_2$ and vice versa, the stronger condition imposed by (2) means that R gives rise to foundation relations in the strong sense under these conditions:

- (44) $R_1 \supset R_2$ iff $\square(\forall x)(x \in R_1 \supset (\exists y)(xRy \& x \prec y \& y \prec x))$
 (45) $R_2 \supset R_1$ iff $\square(\forall x)(x \in R_2 \supset (\exists y)(yRx \& x \prec y \& y \prec x))$

Clearly any relation which is symmetric and for which (44–5) held, would give rise to a derelativised *self-founding* species term: for example from ‘possesses the same parents as and is different from’ we get ‘sibling’ while ‘is working together with’ gives ‘collaborator’.

Certain relative terms which possess etymologically related derelativised nouns fail this test, perhaps rather surprisingly. For example ‘employs’, ‘loves’, ‘shaves’, with their nouns ‘employer’ / ‘employee’, ‘lover’ / ‘loved’ etc. have neither of the cognate pair of nouns founding the other. The reason is that it is possible that all employers, lovers, shavers, etc. employ, love and shave only themselves. In general, so long as a relation *could* be reflexive, even by accident, i. e.

- (46) $\diamond(\forall xy)(xRy \supset x = y)$

then there is no reason why either of R_1 , R_2 should be founded on the other. Of course, in the weaker sense of foundation given by ‘ \supset ’, there is always reciprocal foundation: there can be no employer without an employee, no lover without a loved one etc. But where general reflexivity is possible, this sort of requirement is not a requirement for an *associated* entity as such. It follows that any relative term possessing the logical property of reflexivity, including all equivalence relations, all partial orderings and especially identity, fails to give rise to foundation relations in the strong sense.

One obviously germane relation is the whole-part relation. In fact, if we consider the relation of being a proper part, symbolised ‘ \ll ’, we shall see that this gives rise to one self-founding derelativised term. For ‘ \ll_1 ’ is self-founding, whereas it is not true that $\ll_1 \supset \ll_2$, or that $\ll_2 \supset \ll_1$, or that $\ll_2 \supset \ll_2$. The reason that none of the last three is true is that we can

envisage the situation where the world consists of precisely two atoms, i. e. is a whole with only two proper parts. Again, it is certainly true that $\ll_1 \not\sqsupset \ll_2$ and vice versa, i. e. that there cannot be a proper part unless there is a proper whole or container, and vice versa, but the stronger relation of founding is ruled out by the restrictions of (2) as manifested in (44–5). The obvious noun-phrase corresponding to ‘ \ll_1 ’ is simply ‘proper part’. Because of the mereological law that to every proper part of a whole there must correspond a complementary proper part of that whole, i. e. an object disjoint from it (sharing no parts) which together with it makes up the whole, or, symbolically:

$$(47) \quad \Box(\forall xy)(x \ll y \equiv (\exists z)(x \sqcap z \ \& \ y = x + z))$$

it follows that \ll_1 or ‘proper part’ stands for a self-founding species. It turns out then that even the terms ‘whole’ and ‘part’ are derelativised from one or other of the relations ‘is part of’ or ‘is a proper part of’; this fact leads Husserl into local difficulties in expounding the idea of an independent part, since while it is natural to say that a (proper) part as such cannot exist apart from its whole, for independent parts we also want to say that the object which is here in fact a part *could* exist outside this particular whole.⁸² The difficulty is only one of expression, however, not of substance.

Having seen how foundation relations may arise of relative terms, we might turn the issue round and ask whether all foundation relations point back to some underlying and more basic relative term. The question must first be made more precise however, since for any pair of species α, β such that $\alpha \sqcap \beta$, we always have the relative term ‘ $\alpha \sqcap \beta$ ’. We are trying to get beyond this however and ask whether an α which is founded on a β is so because of some relation which is not defined in terms of α and β , but which may indeed be used in definition of these terms, as in the case of derelativisation already mentioned. If we follow Husserl’s opinion on this, we should have to deny it. For Husserl claims that although *colour* and *extension* are mutually founding, there is nothing in the concepts colour and extension which points to any such underlying relation.⁸³ It is Husserl contends, precisely in this lack of a means to render the law of mutual dependence for colour and extension as an instance of a *logical* or formal principle that there consists the synthetic *a priori* status of the statement that colour is impossible without extension and vice versa. Were it possible to treat ‘colour’ and ‘extension’ as nouns

definable by derelativisation from some antecedently given relative term, the dependence in question would be analytic rather than synthetic. While it seems to me that Husserl's distinction between analytic and synthetic is not so sharp as he thought it was,⁸⁴ the mere possibility that there should be acceptable cases of foundation where the necessity is not obviously logical leaves in doubt the possibility of always finding an underlying relation.

It is worth considering a way of making a distinction among relations which can be found at its clearest perhaps in Meinong, who also brings this distinction into play when discussing the difference between genuine and higher-order objects.⁸⁵ Some relations, such as difference, similarity, being the same height, and the like, do not bring their terms into any real connection, but rather leave them quite unaffected by being thus related. Standing in such relations makes no difference to the properties of the terms; it is indeed often the case that they stand in such a relation in virtue of the separate properties that they possess. Such relations are themselves built or founded on their terms. We may call these *ideal* relations. Other relations, such as acting upon, magnetically attracting, playing tennis against, bring their terms into connection in that, had the relation not obtained, the properties of one or both of the terms would have been different. We may call these *real* relations. The most obvious examples of real relations involve some causal link. Now some foundation relations have underlying relative terms corresponding only to ideal relations, which means that the unity engendered by the foundation is in a sense extrinsic to the objects related. Many ideal relations are equivalence relations, and since these are reflexive they are in any case, by the result above, powerless to engender genuine foundation relations. But where there is some real connection between the terms of a relation, these terms, described in a way which implies the properties induced by the relation, will, if the relation in question satisfies one of (44–5), be foundationally related. We could then describe the relation as a moment of the whole uniting the parts. While these remarks are only schematic, it does seem to me that a theory of the unity of wholes can only be developed in conjunction with an adequate theory of relations: the two enterprises must proceed together. It is perhaps not accidental that the importance of the interconnection between relations and wholes only arises as a serious issue once the Leibnizian dogma that whatever exists is one is called into question.

One of the considerations we derive from examining the role of rela-

tions in engendering foundation is the impoverished role of reflexive relations, including especially equivalence relations. The role of the latter in modern theories of abstraction is well-known. But from the ontological point of view reflexive relations are as such highly dubious. While we may be perfectly prepared to allow a relative *term* to be flanked by a pair of names for the same thing, and yield a true sentence, it is a different matter again if we ask what *relation* corresponds to the term. The whole notion of a relation which holds between a thing and itself is suspect,⁸⁶ and the more especially when, in the case of identity, it can only hold between a thing and itself. This difficulty can be found for instance in Hume and Wittgenstein.⁸⁷ It is usual these days to dismiss their problem as a pseudo-problem resulting from the confusion of a sign with the thing signified.⁸⁸ But the objection is not that there is a certain kind of relative term which can generate true sentences. It is rather that nothing intrinsically relational is represented by this sign, if indeed anything at all is represented. Nor is this to deny the cognitive value of such relative terms. It is to object that they are ontologically sterile. Where a reflexive relation may also hold between different things, as e.g. 'is the same height as', it can always be traded in for the anti-reflexive variant, e.g. 'is the same height as and different from'. Such terms may now generate foundation relations between their derelativisations. Indeed those perplexing derelativisations like 'employer' / 'employee', etc. are most happily applied when reflexivity is not envisaged: it does sound wrong to describe a self-employed person as either an employer or an employee, or a narcissist as a lover, and it is because of such anti-reflexive uses that we have the derelativised nouns at all. It may be of more than etymological interest that many of the terms for equivalence relations are in fact derived from their associated adjectives or nouns, even, it should be noted, *identity*.

Notes

References in these notes are to works listed in the bibliography at the end of these three essays. Works are cited under the name and year in which they appear there.

References to Husserl's *Logische Untersuchungen* (Husserl, 1900–01) will be to the volume and page of the 5th edition of 1968, which will be abbreviated *LU*, and to the page of the English translation of Findlay 1970, abbreviated *LI*. Section numbers, unless otherwise specified, are to the third investigation.

- ¹ An earlier version of this essay was read at the Colloquium 'Whole-Part Theory and the History of Logic' held by the Seminar for Austro-German Philosophy at the University of Sheffield in May 1978. My thanks go especially to David Bell, Kevin Mulligan, Herman Philipse and Barry Smith for their help and constructive criticism.
- ² When William Kneale visited Husserl in Freiburg in January 1928, he relates that Husserl "told me that his essay *Zur Lehre von den Ganzen und Teilen* in his *Log. Unt.* was the best starting point for a study." (From part of a letter to Herbert Spiegelberg quoted in Spiegelberg, 1971, n. 25, p. 78.)
- ³ For Husserl on formal theories see *LU I* §§ 67–72. The ideas are expanded considerably in Husserl, 1929.
- ⁴ The example is Husserl's: *LU II/1* 254, *LI* 457. A purely formal proposition which is true is free of all existential assumptions, § 12 *ibid.*
- ⁵ Wittgenstein, 1961, 4.1272 tells us that words like 'object', 'concept', 'complex' and 'fact' signify formal concepts, and are represented in a *Begriffsschrift* by variables. Cf. also 3.325.
- ⁶ This concept of variability of all propositional constituents except the logical constants can be found already in Bolzano. Cf. his definition of logical analyticity and universal satisfaction in Bolzano, 1837, §§ 147–8, a work which influenced Husserl profoundly.
- ⁷ For explicit repudiations of formalism in mathematics cf. e.g. Husserl, 1929, § 39.
- ⁸ For an introduction to Leśniewski's work see Luschei, 1962 or Lejewski, 1958. For Leśniewski the division between logical and non-logical theories comes between his Ontology and his Mereology, so for him whole-part theory contains non-logical constants. Leonard and Goodman, 1940 or Goodman, 1977 blur such a distinction by defining identity mereologically. For an axiomatisation of the whole-part theory in Goodman, 1977 see Breitkopf, 1978.
- ⁹ *LU II/1* 252, *LI* 455.
- ¹⁰ Such a theory is explicitly canvassed at § 24.
- ¹¹ Aristotle, 1928, Ch. 2, where Aristotle contrasts being part of a subject with being in a subject in such a way as to be incapable of existence apart from it.
- ¹² Stumpf, 1873.
- ¹³ Husserl, 1894.
- ¹⁴ But compare my third essay below, where a nominalistically acceptable conception of set is described.
- ¹⁵ Cf. the discussion in § 18, where Husserl is not altogether clear whether it is possible to give examples of proper parts of a whole which are not proper parts of proper parts of this whole.
- ¹⁶ Wiggins, 1976. Cf. the mereological applications in Wiggins, 1980.
- ¹⁷ Cf. Locke, 1975, Book II, Ch. 27: "In the state of living Creatures, their Identity depends not on a Mass of the same Particles; but on something else. For in them the variation of great parcels of Matter alters not the Identity"; p. 330.
- ¹⁸ Chisholm, 1976, Ch. 3 and appendices A–B.
- ¹⁹ Wiggins, 1980 uses a whole-part theory strengthened with the operator *nec* to argue for this conception and against Chisholm's *entia successiva*.
- ²⁰ Cf. § 12 of the fourth investigation, where it is declared that 'round square' cannot correspond to any object: *LU II/1* 326, *LI* 517. Later, in Husserl, 1948, § 91 the extension of a pure species is said to comprise pure *possibilities*.
- ²¹ § 2, *LU II/1* 252, *LI* 455.
- ²² In commentary on Aristotle, Anscombe in fact replaces Aristotle's accident example by a boundary example: Anscombe and Geach, 1961, pp. 7–8.
- ²³ It is indicative of Husserl's low reliance on categories that he is very reluctant, by comparison with later philosophers, to brand sentences as nonsensical. Cf. his distinction in Investigation IV, § 12, between nonsense and absurdity.

- ²⁴ Husserl, 1948, § 32a. In § 32b Husserl adds *connections* as yet a further distinct kind of dependent part to accidents (which he calls 'qualities') and boundaries.
- ²⁵ § 22, *LUII/1* 280, *LI* 478–9.
- ²⁶ *Ibid.*, *LUII/1* 279, *LI* 477.
- ²⁷ As Husserl says, *LUII/1* 280, *LI* 478, "Unity is... a categorial predicate."
- ²⁸ § 11, *LUII/1* 252, *LI* 455.
- ²⁹ Perhaps the clearest statement of this is in § 148 of Husserl, 1913. Cf. Beilage 74 of the *Husserliana* edition (p. 625), where Husserl clarifies the statement in the text. It will become clear in my third essay below that I do not share Husserl's view that a nominalisation is necessary to constitute a set as a new object on the basis of plural reference.
- ³⁰ Cf. Husserl on *collectiva* at Investigation VI, § 51, *LUII/2* 159, *LI* 798. Though the section title also mentions *disjunctiva* the section has strangely nothing to say about such things.
- ³¹ § 22, *LUII/1* 279–80, *LI* 478.
- ³² § 14, *LUII/1* 261, *LI* 463.
- ³³ § 13, *LUII/1* 258, *LI* 460.
- ³⁴ In § 14 this is seen by the *ad hoc* use of a suffix, in § 13 by the lack of articles. Husserl's usage of symbols is sloppy by modern standards.
- ³⁵ Cf. the remark that we can use the same expressions for individuals and species as a 'harmless equivocation'. § 14, *LUII/1* 261, *LI* 463.
- ³⁶ *Ibid.*
- ³⁷ § 21, *LUII/1* 275, *LI* 475.
- ³⁸ Cf. Proposition 5 of § 14, *LUII/1* 263, *LI* 465.
- ³⁹ This suggestion was made to me by Barry Smith, as an improvement on an earlier formulation of mine which required that α s and β s be such that no α ever be part of a β or vice versa. Both these ideas are inadequate, as the case of self-founding species shows.
- ⁴⁰ § 14, *LUII/1* 261, *LI* 463.
- ⁴¹ § 21, *LUII/1* 276, *LI* 475.
- ⁴² On entailment see above all Anderson and Belnap 1975.
- ⁴³ § 14, *LUII/1* 262, *LI* 463.
- ⁴⁴ This is one place where the definition of foundation using entailment suggested at (4) is too strong for our purposes, because the definition of ' α ' β ' is extensional while that of ' \neg ' in (4) is relevant and intensional, and the restrictions in E on e. g. importation make it impossible to prove (7) as it stands. However by strengthening the definition (6) a relevant version of (7) could be proved.
- ⁴⁵ § 16.
- ⁴⁶ Herman Philipse has objected that the husband/wife type examples are analytic, whereas Husserl is clearly interested in synthetic connections such as the colour/extension example. Two things may be said in reply. Firstly, the distinction Husserl draws between analytic and synthetic is not as sharp as he thought it was. This is an issue which I hope to take up elsewhere, though note the remarks in n. 77 of the opening essay by Smith and Mulligan. Secondly, as Husserl is really interested in an *a priori* theory (§ 24) no harm at all can be done by including analytic as well as synthetic examples.
- ⁴⁷ Ginsberg, 1929. Cf. my note to the translation of her later paper in this volume.
- ⁴⁸ Cf. Husserl, 1913, § 14.
- ⁴⁹ *Ibid.*, § 12.
- ⁵⁰ For an implicit recognition of this in Leibniz, cf. Leibniz 1903, 261. Cf. more explicitly Ishiguro, 1972, 16.
- ⁵¹ I had previously thought that it made some sense to talk of individuals simply as such, without mention or assumption of any kind to which they might belong. Many sources have dissuaded me of this view, but David Bell and Herman Philipse have done so most directly.

- ⁵² Aquinas, 1964–76, Ia, 44, 4; Vol. 8, p. 21, where Aquinas argues that in creating God does not act from need but ‘simply to give of his goodness’. At the same time the idea that God’s nature could have been other than it is not particularly congenial for Aquinas, so his problem is not completely cleared.
- ⁵³ Ingarden, 1964–5, §§ 12–15.
- ⁵⁴ For a note on these translations cf. my introduction to Ginsberg’s paper.
- ⁵⁵ *Ibid.* § 12.
- ⁵⁶ *Ibid.* § 13.
- ⁵⁷ *Ibid.* § 14.
- ⁵⁸ *Ibid.* § 15.
- ⁵⁹ One of Ingarden’s examples of a heteronomous object would be any entity which is purely noematic, a correlate of consciousness, so Ingarden could fairly claim that the material for such a distinction exists already in Husserl.
- ⁶⁰ Chisholm, 1976, 208. This paper in general furnishes abundant evidence that late Brentano was working using whole-part theoretic considerations akin to those we find in Husserl, which is not surprising, given his influences on his pupils, in particular Husserl’s former teacher Stumpf. Unfortunately we have not here the space to compare Husserl and Brentano at length.
- ⁶¹ The example, though not the application of it, is drawn from Ingarden, 1964/5, § 15. The idea of disjunctive satisfaction of requirements clearly has applications in biology. The importance of such biological considerations is urged in the Preface to Wiggins, 1967.
- ⁶² Because Chisholm denies that a genuine entity may lose parts, he must construe organisms and machines as less than genuine, with an identity which is a simulacrum of true identity. Cf. Chisholm, 1976. This is a thought which can be found *inter alia* in Hume and Leibniz, and in a modified form pervades extensional mereology. It is certainly attractive, and more tractable than the Aristotelian alternative, but I am convinced it is wrong.
- ⁶³ § 13. *LU* II/1 258, *LI* 460.
- ⁶⁴ Ginsberg 1929, 112. Cf. also my prefatory note to her paper in this volume.
- ⁶⁵ Wiggins uses this consideration in his 1980 to discredit the idea that a cat can be identical with the mereological sum of its body + its tail, for the cat, but not the sum, could lose the tail.
- ⁶⁶ As suggested in Kripke, 1972, 312 f.
- ⁶⁷ This notation for the proper ancestral is due to Carnap. Cf. his 1954, § 36.
- ⁶⁸ § 14, *LU* II/1 262, *LI* 464.
- ⁶⁹ On Frege’s use of whole-part terminology to describe the phenomenon which he calls ‘unsaturatedness’ of predicates and concepts see his late essays “Die Verneinung” and “Gedankengefüge”, Frege, 1976a. I have expanded elsewhere on the appropriateness of using Husserlian ideas in this connection. Frege’s use of terms like ‘ergänzungsbedürftig’, unlike that of Husserl, is not backed by a theory of dependent and independent parts. Indeed, if we are to believe his remarks in Frege, 1895, Frege had a rather low opinion of whole-part theory in general.
- ⁷⁰ § 14, *LU* II/1 263, *LI* 464. We have adjusted the symbolism to our convention.
- ⁷¹ Ginsberg, 1929.
- ⁷² Cf. Meinong, 1899.
- ⁷³ On categorial unities see § 23, for instance. The notion can be found throughout Husserl’s writings.
- ⁷⁴ Ingarden, 1964/5, § 15, n.
- ⁷⁵ Findlay, 1963, 148f.
- ⁷⁶ The idea of separate presentation here derives from Stumpf, 1873. Cf. § 3 for Husserl’s comments, and the historical remarks in Ginsberg’s paper in this volume.

⁷⁷ §25, *LU* II/1 292, *LI* 488.

⁷⁸ Findlay, 1963, 150.

⁷⁹ The Prague and Moscow schools of linguistics were in fact influenced by the third investigation. Cf. Jakobson, 1973, 13–4, Holenstein, 1975.

⁸⁰ Cf. Pantin, 1968, 37f.

⁸¹ Aristotle, 1930, 292^a 19f.

⁸² § 11, *LU* II/1 253, *LI* 456.

⁸³ *Ibid.*

⁸⁴ As mentioned in n. 46 above.

⁸⁵ Cf. Findlay, 1963, 141f. Husserl in fact makes a very similar distinction between two sorts of relation in Husserl, 1887 and 1891a. His terminology is however more unfortunate, since he calls the relations 'physical' and 'psychical' rather than 'real' and 'ideal'. The term 'psychical' indicates not that the relation is mental, but that it is *of a sort with* the relation between object and content of an idea. Cf. Findlay, 1963, 35, where it is made clear that the mental relation is ideal for Meinong. The terminology of Husserl readily misled Frege into criticising Husserl's theory of number as psychologistic, which it was not. For a clear refutation of the myth of Husserl's early psychologism see Willard, 1974. We have, for obvious reasons, adopted the less misleading terminology of Meinong.

⁸⁶ Cf. the arguments against reflexive relatedness in Vol. 2 of Armstrong, 1978, 91f.

⁸⁷ Hume, 1978, Book I, Pt. IV, § II, p. 200f. Wittgenstein, 1961, 5.53f.

⁸⁸ Cf. Quine, 1960, § 24.

II. Number and Manifolds

§ 1 Introduction: The Philosophy of Number

An adequate philosophical theory of whole numbers has to be able both to give an account of what we accomplish when we make empirical ascriptions of number, for example in answer to “How many . . . ?” questions, and also to provide an account of the content and validity of the propositions of arithmetic. I shall call the theory surrounding the first kind of question the philosophy of *number*, and that surrounding the second kind the philosophy of *arithmetic*. An adequate account must further provide some explanation of the link between the two. As attempts at such philosophical accounts one may take the different philosophies of Frege and Husserl. Frege explained ascriptions of number as assertions about a concept, while he explained arithmetical propositions as concerning certain abstract objects, the whole numbers themselves. The link between the two accounts is provided by his theory of abstraction, in which expressions like ‘the number 0’ are contextually defined. Husserl’s theory on the other hand takes ascriptions of number to concern not concepts but totalities, which are given to the mind in an act of “collective combination”. The transition to abstract numbers is accomplished by a theory of symbolisation, according to which larger numbers are presented to us mediately, through numerical expressions. While there are difficulties in both accounts, they have in common that they present a two-stage theory of number, the first stage dealing with empirical ascriptions of number, the second with the formal validity of arithmetic, with a bridge between the two consisting of a theory of abstraction. Subsequently, interest has shifted almost exclusively towards the philosophy of arithmetic, with various attempts being made, in the wake of Frege’s unsuccessful one, to provide a basis for arithmetic in a formal theory such as the typed logic of Whitehead and Russell or an axiomatic set theory. With this emphasis on deriving arithmetic from some more general theory has gone an increasing willingness, evident already to some extent in Frege, to let numbers be any handy construction with the right formal properties. Since there are many such constructions available, this has led to scepticism that there are such entities as num-

bers at all, over and above the many series of *numerals*, which are expressions suitable for performing a count.¹ I do not share such scepticism.

§ 2 Frege's Criticisms of Manifold Theory

To counteract the tendency to concentrate on arithmetic and ignore the existence of empirical ascriptions of number, I wish to suggest in outline the form which an adequate philosophy of *number* should take. Obviously this cannot be done without an eye as to the likely way in which we arrive at arithmetic, but I shall not stress here the role of a theory of abstraction in any detail. These remarks indicate that I am convinced that a two-stage theory of whole numbers is correct, and I shall be here attempting to say what the first stage should consist in. I maintain that, in a sense to be explained fully below, number is a property of external things of a kind which I call *manifolds*.² In this I shall basically agree with Husserl against Frege, but the theory involved will perforce take account of Frege's objections to such a theory. These objections are the following:

- (1) that if we try to ascribe number to external things, we find we cannot do so consistently, because one and the same thing may be ascribed many different numerical predicates: a pack of cards may number one (pack), or 52 (cards), or 4 (suits) and so on.
- (2) that there is nothing to which we can ascribe the number 0 on such a theory.
- (3) that number is very different from all other properties of external things.
- (4) that number cannot be identified with the way in which a thing may be split up into parts.
- (5) that things do not need to be literally collected together in order to be numbered.
- (6) that the concept of number has a far wider range than the concept of physical thing: we may apply number universally, to non-sensible and abstract things such as the figures of the syllogism as well as to concrete physical things such as boots.³

In defending a theory of number as (in the first stage) a property of external things, I shall take account of all these objections in one way or an-

other. While I shall agree with Frege over (4) I shall nevertheless suggest that there is, for many manifolds, a quite close connection between their numerical properties and their mereological properties, so that my account will be in part a *mereological* theory of number.

My account of number then stays fairly close to that of Husserl, who is by far the most sophisticated “external thing” theorist to date. His theory has answers to all of Frege’s objections, but not all these answers are of equal quality or acceptability, e.g. his rejection of 1, as well as 0, as a genuine number. Also, while Husserl approaches number through a consideration of mental acts, ours takes much more linguistic considerations as starting point. So e.g. Husserl’s answer to objection (6) is that collective mental acts may consider together objects of any category, while ours is that plural referring expressions may be put together out of terms referring to objects in any category. This difference of starting point by no means rules out the possibility of a more complete unified treatment in which both language and mental acts have their proper place: it is indeed a longer-term desideratum. But my chief disagreement with Husserl is over his contention that pluralities are *constituted* as such by acts of collective combination, and accordingly are higher-order, *categorial* objects. I hold that manifolds are lower-order multiplicities rather than higher-order unities, and that Husserl was here under the pervasive influence of the prejudice in favour of the singular, in a weak but crucial form: weak, because Husserl accepts that we may have mental acts simultaneously directed to many objects at once, but regards number and manifolds as being first constituted in a higher-order act reflecting on such plural consciousness, a move which I hold to be superfluous once the distinctive nature of pluralities is recognised.

One very general form in which we may describe the (possibly unknown) number belonging to a given totality is

the number of *cs* which Φ

where *c* is a common count noun or count noun phrase,⁴ and Φ is an intransitive verb or intransitive verb phrase. For example we have

the number of men in the Red Army

the number of trees in Sherwood Forest

the number of women who have had more than ten children.

Two important variable features of this form are the count noun (phrase) *c* and the intransitive verb (phrase) Φ . Both contribute to deter-

mining what the answer to the question 'How many *cs* are there which Φ ?' The noun tells us what *sort* of thing we have to count: it supplies what I shall call the *counting principle*. In the above examples men, trees and women are what is counted respectively. The verb tells us not what sort of things we should count, but which things among those of that sort given by the noun. It sets limits on the ones which 'count', so I shall say it provides the *delimiting principle*. I distinguish three kinds of delimiting principles:

- (1) restrictive
- (2) limitative
- (3) mixed.

Restrictive principles limit the *cs* which matter to those which possess a certain property, or stand in a certain relation, and so on, where no reference is made to spatio-temporal position. For instance we get the following noun-phrases by restricting a noun in this fashion:

tree which is taller than 50 metres
man who admires Cleopatra
woman who has had ten or more children.

Limitative principles set spatio-temporal limits within which the counted *cs* must fall, such as

Public House within two miles of Trafalgar Square
woman in England on January 1st 1979

and *mixed principles*, as their name implies, are neither purely restrictive nor purely limitative, for example

woman in England of January 1st 1979 who has had ten or more children.

It seems plausible that mixed principles are the most common.

Frege called count common noun phrases of the kind we have been considering *concept-words*. His use of the term 'concept' was and is deviant in two respects. Firstly, whereas traditionally concepts would have been regarded as the *senses* of predicate or noun-expressions, for Frege they were their *referents*. Frege was aware of this and pointed it out in correspondence with Husserl.⁵ Secondly, since Frege intended to interpret numbers as 'concerning' concepts, he allowed expressions with de-

limiting clauses to stand for concepts. Those which had variable aspects, such as a tensed verb, could have different numbers at different times, but the clearest cases were those which contained no variable elements, whose number 'is the same for all eternity'.⁶

Now the second point, that Frege extended the term 'concept' to include those concepts belonging to expressions containing delimiting phrases, as in our examples, does not appear to me to be a serious one. There are precedents for even unitarily-lexicalized expressions involving some reference to particular individuals, or to some place or time, e.g. 'Aristotelian', 'mediaeval', 'Scandinavian'. The first point, that it would be happier to call the senses of predicate or noun-expressions 'concepts' seems to me, at least so far as it is a dispute about terminology, to be one where it is better to side against Frege for the sake of clarity. This does not prejudice one's attitude to Frege's position about functions and objects, since one could simply adopt a new word for those functions which take objects as arguments and yield truth-values as values, instead of Frege's 'concept'. Since however I have no intention of defending a Fregean ontology or philosophy of number, but wish to provide an alternative, it is worth pointing out that Frege wanted concepts to be something objective, so that numerical predications could be objectively true. But this objectivity would equally be guaranteed were we to interpret concepts as senses. The role of the concept was to unify or collect individuals in a way which does not involve physical displacement. Now of course no literal collection goes on at all here, so it would be better to replace talk about collection and unification altogether. The role of concepts (in the traditional sense) in enabling us to refer to manifolds will be set out below.

§ 3 Plural Terms and their Designata: Manifolds

Frege used the term *Eigenname* for both proper names like 'Aristotle' and definite descriptions. These have often been classed together as singular *terms*.⁷ The rationale for this is that both kinds of expression perform a similar role: that of making a definite reference to something. Of course they work in different ways, but this does not make it appropriate to put them into completely different categories. They are syntactically intersubstitutable. This is not sufficient to mark out the category of *terms*, which I shall be interested in, since terms are also intersubstitut-

ible *salva congruitate* with *quantifier phrases* such as 'some man', 'several days', 'all rabbits' etc.⁸ These do not serve to make reference to things, as can be seen by their different behaviour, in connection with negation, from terms.⁹ I wish to consider among terms two other great categories of expression: firstly those which refer only in a specific context, so-called *indexical expressions*. These are a rag-bag of assorted expressions which share only this feature of reference only within a concrete context, and include personal pronouns and expressions with demonstrative pronouns, e.g. 'we', 'that man', 'my friend', etc. I am not particularly interested in such terms in the present context, and include them mainly for the sake of completeness. Much more important for present purposes are *plural terms*: the sort of expression which can be used to refer to more than one thing at once, e.g. 'my friends', 'the men in this room', 'Jack and Jill'. Plural terms are the Cinderellas of philosophical grammar, and very few philosophers have recognized them. One notable exception is the early Russell.¹⁰ Otherwise there has subsisted a remarkable prejudice in favour of the singular, which has not been without its deleterious effect on the philosophy of number.

Plural terms comprise plural descriptions, plural proper names, if there are any, and conjunctive lists of terms, singular and plural. Here are samples of various kinds of plural term to set alongside the more familiar kinds of singular term.

Plural proper name	Benelux
Plural definite description	the fishermen of England
Plural demonstrative	these books
Plural personal pronoun	they
Name list	Tom, Dick and Harry
Mixed term list	Jason and the Argonauts

Whether an expression actually manages to designate something on a particular occasion of its use is immaterial to its status of being a term. For instance the expression 'the greatest prime number' necessarily does not designate anything, but we can recognize it as a term in virtue of its syntactic structure and the syntactic categories of its components: it comes from the same syntactic bag as an expression like 'the tallest man in Finland', and the expressions 'greatest' and 'prime number' themselves occur within terms which may designate something.

Since I take number to be a property of manifolds, and manifolds to stand to plural terms as individuals stand to singular, it will be very im-

portant to get clear how plural reference works. We might say that 'manifold' is the plural of 'individual'. Whether or not a term actually picks out a manifold on a particular occasion of its use is, as with terms in general, not the main issue. Just as individuals are what *could be* designated by a singular term, so manifolds are what *could be* designated by a plural term. Just as one and the same expression which is a singular term may on different occasions of its use denote different individuals, so one plural term may also on different occasions designate different manifolds. Just as 'the President of the United States' denotes different men at different times, so 'Farmer Brown's prize herd of Friesians' may on different occasions designate different manifolds of beasts. Similarly, just as two terms with different meanings may yet have the same referent when singular, so two plural terms with different meanings may yet have the same referents. For instance, if the men in a certain car are Tom, Dick and Harry Jones, the sons of Donald and Edna, the following three plural terms may all designate the same three men:

Tom, Dick and Harry
the men in the car
the sons of Donald and Edna Jones

Hence Frege's sense/reference distinction carries across without difficulty to plural terms.

For an expression to designate a manifold is simply for it to designate each of a number of individuals. There is no difference between the manifold, and the several individuals, despite the fact that we can talk about *a* manifold, and indeed can count manifolds to some extent as though they were individuals. So when an expression designates A and B and C . . . , where these are individuals, this is to say no more than that it designates A and designates B and designates C . . . Russell at one time thought that he could discern these two ways of "denoting",¹¹ but the attempt to do so landed him in the most dreadful muddles about one and many. There is indeed a genuine one/many problem to be laid to rest here, and Russell is not to be deprecated for appreciating this. Commenting on Russell's problem, Quine finds it difficult to see why there should be any difficulty in a set's having many members, any more than there is a difficulty in a single attribute's applying to many things.¹² But Russell's problem was not how something may have many things related to it, e.g. how a man may have many brothers. It is rather the problem of how one thing can also *be* many. It is Quine's insouciance rather than

Russell's difficulty which I find appalling here. I shall return to the one/multi question later in discussing whether there can be manifolds of manifolds.

It is not too surprising that Russell's view of denoting as in every case being the same kind of relation, coupled with the view that even quantifier phrases denote, should give rise to difficulties. It might be objected however that in *my* use of the idea I have gone too far: while it may be acceptable to say that 'the man in this room' denotes Henry on a certain occasion, surely it can't be true that 'the men in this room' could denote John and also denote Henry? Surely it would then be ambiguous – which one does it denote? Ought we not to consider either that it denotes some third thing, such as a class, or that it had better be paraphrased away in favour of singular terms only? This attitude shows clearly the hold of the prejudice in favour of the singular. While in many contexts plurals can be paraphrased away, it is not certain that this applies in all cases, and furthermore there appears to be no good reason to seek such a paraphrase, once the harmlessness of plural terms has been recognized. There is further no reason to suppose that all the familiar properties of singular terms carry across to plural terms. If it is objected that a term cannot denote two things equally without ambiguity, since denoting *means* denoting one thing, then I can simply hand the objector the term 'denote' and set aside 'designate' for my own ends. Though I do not believe that 'denote', especially in the liberal hands of Russell, started out as being confined to singular denotation, it has perhaps, through familiarity, come to have that connotation, and I shall for the sake of clarity, distinguish between denoting, which is the special case of a term's designating one individual, and designating in general. Now a singular term, that is a term which is syntactically singular and is not a collective term, is indeed defective if it could equally well denote two individuals. In such a case we are either charitable about the grammatical number or are strict and declare the term thereby empty. In a converse fashion, a plural term which fails to designate more than one thing ought strictly to be taken as empty, though frequently charity prevails, where there is just one individual which, but for the difference in grammatical number, would be denoted by the term.

The plausibility of taking a plural term to designate each of the several things it does, appears to vary according to the kind of term involved. Let us see how this is so. With any compound term there may or may not be components which can stand on their own as terms. This is already

apparent in singular descriptions, e.g. 'the present King of France' contains the term 'France'. I shall call these *subterms* of the term in question. The same applies, even more so, to plural terms. Any plural term obtained by conjunctively listing other terms obviously contains some subterms. Let any individual denoted by a subterm be said to be *subdenoted* by the whole term, and let any individual designated by a subterm be said to be *subdesignated* by the whole term. Thus e.g. 'John and Henry' subdenotes John and also Henry, while 'Jason and the Argonauts' would subdenote Jason and subdesignate each argonaut, if there were any.¹³ Something may be subdesignated by a term and designated by it also, as the above examples show. More surprisingly, perhaps, one individual may be both denoted and subdenoted by the same term, as e.g. Arthur may be by the term 'Arthur's favourite person' (since he happens to be a narcissist). An individual may be designated without being subdesignated, and vice versa, as a term like 'Arthur's mother' can show. An individual may be denoted or designated, subdenoted or subdesignated more than once by a term. The idea that plural designation is more plausible in some cases than others stems from the differences between those cases where the individuals designated by a plural term are also subdenoted by it, and those where this does not happen. So e.g. 'John and Henry' both designates and subdenotes John and Henry, whereas 'the men in this room', if it designates them, subdenotes neither and subdesignates neither, while it subdenotes this room. But once again, a liking for subdenotation seems to be simply a further manifestation of a preference for singular terms. Certainly a plural term doesn't designate *just one* individual – but then if it did it would not be a plural term! In the case in question, John and Henry both bear the same kind of relation to the term 'the men in this room': it is simply this that I am calling 'designation'.

§ 4 Against the Group Theory of Number

I shall return later to a discussion of what sorts of term designate manifolds under what conditions. First I wish to reject two other possible candidates which have been suggested as bearers of number-properties, but which, unlike Frege's 'concepts', are also 'external things'.¹⁴ I shall call these, respectively, *aggregates* and *groups*. *Aggregates* are defined mereologically. The aggregate of A and B is that individual all of whose

parts have some part in common with A or B; the aggregate of the *cs* is that individual all of whose parts have some part in common with at least one *c*. In general, the aggregate of a number of individuals is the smallest individual all of whose parts overlap some of those individuals.¹⁵ There are attractions to having aggregates as bearers of number-properties. Firstly they are reassuringly concrete. The aggregate of a number of things which take up space itself takes up space, just the space they take up. Secondly they provide a unitary bearer for the property. It is no detriment to an individual's status as such that it has parts discontinuous from one another. Denmark is no less a country than any other for being rather scattered. Certainly some aggregates are more natural than others: an organism is much more a whole than the aggregate consisting of my pet hamster's left ear and the Isle of Mull. It would however be unwise to rule out such bizarre individuals simply because they are bizarre. In bizarre circumstances there could be reason to regard the aggregate as less unnatural, e.g. if I owned the island as well as the hamster and left the island with the hamster's left ear as a bequest. The aggregate would even have a market value, should anyone feel like buying it from me. The reason for liberality in admitting individuals is not the capricious one of giving lawyers or philosophers silly tasks, however. It is because we have no clear idea where to draw a line between natural and unnatural wholes that we cannot afford to be too dogmatic about what is to count as an individual: the bizarre aggregates are a whimsical side-effect of this conceptual caution. So, since all aggregates are unitary, though not all naturally so, aggregates provide unitary bearers for number-properties.

This however leaves the original objection of Frege against "external things" unanswered. One and the same aggregate may have many different numbers ascribed to it: the pack of cards example will serve. It is one pack, but also the aggregate of fifty-two cards, etc. It is for this reason that Russell, for instance, took numbers to be properties of classes, rather than what he called "wholes", since a whole is essentially one rather than many.¹⁶ Strictly, this does not rule aggregates out as bearers of number-properties, but it does rule them out as bearers of number-properties other than the first, which would cripple a theory of number so based.

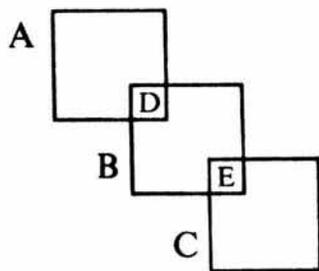
However, it may be objected that Frege's position depends upon a hidden assumption which may itself be questioned, namely that different number-properties are mutually exclusive, so that one thing cannot

have more than one number correctly ascribed to it. Armstrong has recently defended the view that one particular may be ascribable many numbers.¹⁷ The pack of cards has the formal properties of being fifty-two-parted (having fifty-two parts), being four-parted etc. Whatever is n -parted is also m -parted, where $m < n$. To see this, we can take the original n parts and consider $(m-1)$ of them left unaltered, taken together with the remaining $(n-m+1)$ aggregated as a single part. Armstrong takes this relation between these formal properties itself to be a part-whole relation. The difference between the pack as fifty-two cards and as four suits is easy to account for, since each of the requisite parts falls under the predicate 'is a card', while no proper part of it does, and it is not a proper part of anything that does either. Armstrong sketches the transition to arithmetic by taking the numbers to be the logically possible set of properties, *being-two-parted*, *being three-parted*, . . . etc. Number, for Armstrong, attaches to a class as one, i.e. an aggregate, rather than a class as many. But number attaches to the aggregate not merely taken as a heap, but taken as exemplifying certain properties which divide it into parts.

Now it seems to me that, while there is much to be said for this view, in particular that Armstrong recognises the importance of both mereological considerations and plural reference, it cannot be the final word on the subject. Frege's objection is based on the view that, at some level, number-predicates are mutually exclusive, and he is surely right on this, otherwise there would be no correct answers to "How many . . . ?" questions, or rather, there would be many correct answers, and this does not accord with our practice. It looks as though Armstrong may avoid this problem by having recourse to the case not of formal properties like *being three-parted*, but material number-properties like *being an aggregate of three apples*. Nothing can at the same time be an aggregate of three apples and an aggregate of some other number of apples. This sort of consideration, which obviously relates closely to the sortal noun 'apple', is presumably what facilitates our normal practice and prevents us from having to ascribe different number-predicates to one and the same aggregate: we think of it not merely as an aggregate, but as an aggregate of cs , where c goes proxy for some suitable common noun (phrase). This accounts for the necessity of importing a common noun, what Frege called a concept-word.

However this account still does not separate the number-properties so that they are mutually exclusive. In the following figure, let 'square'

mean 'area bounded by and including a square figure'. Then the figure as a whole



may be taken as the aggregate of three, four or five squares, that is, either as $A + B + C$, or $A + B + C + D$, or $A + B + C + E$, or $A + B + C + D + E$, where '+' denotes mereological fusion or summation. The reason is that a square, unlike an apple, can be a proper part of one of its own kind.

For this reason we might try a second, related suggestion, that numbers attach not to aggregates *per se*, but to *groups*, which are aggregates *qua* composed in a certain way. The number three attaches to the above aggregate only *qua* the aggregate of A, B and C, and not *qua* the aggregate of A, B, C and D, etc. Number-properties are then not categorematic, but syncategorematic. What is true of an aggregate *qua* composed in one way need not be true of it *qua* composed in another. This version of the concept of group I have taken from Sprigge,¹⁸ who derives the word from McTaggart.¹⁹ McTaggart seems to me to use the word much more as I would use the word 'manifold', since while he allows that two groups may have what he calls the same 'content', as e.g. the groups A, B, C and A, B, C, D do, he regards them still as different groups, since they have different members. Groups are determined by their members for McTaggart, so I think that there is reason to suppose that *his* group is a class taken in extension, or what I call a manifold.²⁰ Sprigge on the other hand regards two different groups with the same content as absolutely identical, though not the same group. This is because the group is simply the aggregate *qua* composed thus and so, whereas 'is the same group as' is not for Sprigge a genuinely relational predicate. 'The group of F's is the same group as the group of G's' means for Sprigge 'The aggregate of F's *qua* being the aggregate of F's has the same members as the group of G's'. Now there is no need to relegate the predicate 'is the same group as' to such a lowly status if one accepts, as Sprigge does not, that identity

is relative, or at least that there can be relative identity predicates, of which 'is the same group as' and 'is the same aggregate as' are two. One and the same entity may be the same aggregate as something but not the same group as it. The aggregate is then the same aggregate as, but a different group from, other groups with the same underlying content.

I wish, so far as is here possible, to skirt the vexed question of relative identity, which obviously bears on our issues quite closely, but which can I think be partly set aside at least in regard to the problem of what it is that bears number-properties.²¹ For whether we take Sprigge's view of groups, or a relativist view, there is still, I believe, a serious objection to taking groups as the bearers of such properties. This is that there are certain pluralities which can never form groups, since their members could never be considered to compose an aggregate. Obviously anyone who is unhappy with the "bizarre" aggregates previously mentioned will feel this objection to be even stronger than I find it. Even if we are happy with aggregating entities of different categories, e.g. events and continuants, as e.g. when we consider that aggregate which is the sum of a man and all the events befalling him in his life, some cross-categorical aggregates are just too incredible. A sigh, Chairman Mao and the number five could never be considered to form a whole, if only because the senses in which we talk about their *parts* differs from one to the other. They can of course be *considered together*, as we have just done, and they may be the extension of the concept 'things I have thought about in the last minute': to *that* extent they may be unified. But this unification is purely extrinsic to them, and lies in the acts of mind of the reader, and in the occurrence together of expressions denoting them within the compass of a single term, or in their all satisfying a certain predicate. I confess to being unable to decide where to draw the line between merely bizarre aggregates and pluralities to which no aggregate corresponds. However, there is a still stronger reason for regarding some pluralities as unable to form an aggregate. The examples I have in mind are those where we have to do with a number of mutually exclusive possible states of affairs. Suppose, for instance that I wish to calculate how many different ways the first two cards in a pack may turn up when I next deal. Then since the cards will, on this next deal, only turn up in one of these ways, to the exclusion of all the others, I cannot be counting actual events, since the same answer holds whether I run through the gamut or not. Even if I do run through all the different ways, I cannot run through them all on the *next* deal, since this can occur only once. So what is counted or calculated

must be a number of possibilities, possible events, say. Each such event, were it to happen, would thereby exclude all the others from happening. Now these many mutually exclusive possible kinds of event could never form a whole, since they are all, so to speak, denizens of different possible worlds. It would be a horrendous ontology which allowed aggregates composed of events in as many different possible worlds to form an aggregate across worlds, even supposing one were happy with this view of possibility anyway. The mutually exclusive possibilities can of course be *considered together*, but this is something quite different from their forming a whole. The very least we can ask of an aggregate is that its parts can mutually co-exist in one world, even if they do not all exist at the same time. This requirement also would disallow aggregates composed of parts from different ontological realms, if there is more than one, e.g. a man and a Platonic universal. It is noticeable that the defenders of aggregates and groups tend to pick their examples from a single category and a single realm. Armstrong, for instance, is a physicalist, so there is some plausibility in his espousal of aggregates as bearers of number-properties. I do not wish my theory of number to be so tied to a particular ontological doctrine, so that if the ontology fails, so does the theory of number. This is a good reason for avoiding Frege's view of numbers as the properties of concepts (at least for the first stage of a philosophy of number): Frege's objective concepts, referents of predicate-expressions, are ontologically dubious.²²

The aggregate and group are attractive as bearers of number-properties because they are unitary. This is an aspect of the prejudice in favour of the singular: it is deemed that whatever has a property must be *one* thing, so whatever has number-properties must also, in some sense, be one thing. It seems to me, on the contrary, that some properties of their very nature are borne by more than one thing. This is, I think, Armstrong's reason for ascribing number-properties to aggregates rather than manifolds, even though he is aware that plural reference may make possible a non-platonistic theory of classes. This is because Armstrong believes plural reference to be essentially eliminable: where it is easily eliminable, as in

Tom, Dick and Harry went to a party,

in favour of a triple conjunction, Armstrong is happy, but where elimination is not straightforward, as in

Tom, Dick and Harry lifted a girder,

Armstrong balks at plural reference, and prefers to say that the plural term refers to a single entity, namely the girder-lifting *team*.²³ My point, to which I shall return later, is that a word like 'team' is itself already a collective noun, which may be true of a plurality of individuals without being true of the individual members. Armstrong shows that he accepts plural reference as such only where the predicate satisfied by the entities designated by the plural subject is satisfied by the entities separately. Such a case I shall call *perfect distribution* of the predicate.²⁴ There are many predicates, of which numerical ones offer a clear example, which are not thus perfectly distributive, e.g. 'played a competitive game of chess', 'can speak seventeen languages (between them)'. It may be that plural reference is eliminable in these cases. In the case of number-properties I am not so sure. In any case, eliminability is not in itself something to be held against a certain kind of expression.

We have grown accustomed to the idea of relations being true of more than one thing, and not wishing to reduce relational predications to attributive. The same considerations apply, *mutatis mutandis*, to predicates having plural subjects. Some of them may find their foundation in singulars, just as some relational predications are true in virtue of properties of the individuals involved. There is in fact little to choose in many instances between describing something as a relation holding among n different individuals and as a property of these n individuals which does not distribute to them or to any submanifold of them. This is so where the relation contains no asymmetry, i.e. if ' $Ra_1 \dots a_n$ ' is true, then so is the predication obtained by permuting the terms a_i in any order. This is shown even in ordinary English, where symmetrical relations are as often as not expressed with a plural, conjoined subject, e.g. We may express the predicate ' ξ is playing chess against ζ ', which is symmetric, by the predicate ' ξ and ζ are playing chess', or we may say 'John and Henry are the same age', 'The Jones brothers all sleep in one bed'. The last example shows an aspect of the utility of plural reference: it remains true and expressible even when we don't know how many Jones brothers there are.

We may indeed have arrived at the numerical properties simply by turning certain relational predications into the requisite form with a plural subject. 'A is different from B' is certainly logically equivalent to 'A and B are two (different things)', so we may proceed to 'A is different

from B, and B from C, and C from A' for 'A, B and C are three', etc. We get the general *form* of such predicates by considering the variable predicate '... are all different'. This may take plural subjects designating any number of individuals from two upwards: how many individuals are so designated determines which of the infinitely many number properties is picked out. This account presupposes the applicability of the concept of absolute identity. For those unhappy with this, and for those who believe that there are additional relative identity predicates, we may offer a sortalised version:

'A is a different *c* from B' for 'A and B are two *cs*'

'A is a different *c* from B, and B from C, and C from A' for 'A, B and C are three *cs*'.

Here it is assumed that the truth of 'A is a different *c* from B' ensures the truth of both 'A is a *c*' and 'B is a *c*'. As before, we have the general form of the sortalised numerical predicate: '... are all different *cs*'. It seems to me that there is nothing to choose between seeing the numerical predicate as true in virtue of a multiply-adic relation among individuals, and true in virtue of a non-distributive property of a manifold, save that in the latter case we have a general recipe for constructing one predicate from the next, namely lengthening the conjunctive subject by one sub-term.

This then accounts for the ubiquity of number. It is as ubiquitous as identity and difference.²⁵ Manifolds, that is, 'more-than-ones', are just as "external" as individuals. They do not need to be regarded as having a unity which is "just so much ... as is required to make them many, and not enough to prevent them from being many". The unity of a manifold is no less than that of an individual: it consists in the manifold's being *just these* individuals and no others. Of course it may require a feat of mind to *consider* the many things together, especially if there are rather many of them and we have no handy predicate true just of them. But it may equally require a feat of mind to think of certain individuals. They are no less 'external' and objective for that. We may have access to certain manifolds only because we have plural expressions which we can concoct. But the same problem of remoteness affects individuals: some of these we can only consider through the offices of language. But considering and being are two different things. A manifold exists in so far as its members exist: it is 'external' and 'objective' in so far as its members are.

It is worth pausing at this stage to consider how far we have answered Frege's objections to an 'external things' theory of number. The first problem has been successfully taken care of. Frege's objection applies only to a crass view of external things according to which they are all aggregates, heaps of matter or 'chunks of reality'. What is the relation between aggregates and manifolds on my account? A manifold has the same kind of being as its members. If these are homogeneous enough to be able to constitute an aggregate, then the relation between the aggregate of A_1, A_2, \dots and A_1, A_2, \dots is that they *compose* it, since if we call the aggregate A , we have that A is identical with $A_1 + A_2 + \dots$, the mereological sum of the many individuals in the manifold. The manifold and its aggregate *coincide*, that is, they occupy the same spatio-temporal region, but they are different, for the aggregate is one, and the manifold is many. The aggregate could be referred to by a singular term which does not presuppose plural reference, whereas the manifold must be referred to plurally. The aggregate has the numerical properties *indirectly*, through being the fusion of the members of manifolds which have them directly. There is no problem at this level of the numerical predicates being mutually exclusive: one and the same thing may be composed of three squares as well as of five squares. However, the same manifold cannot at the same time *be* three squares and *be* five squares: Frege's insight that the numerical predicates are, strictly interpreted, mutually exclusive, is upheld. The problem of the number zero has not yet been solved. I shall return to it, and also the number one, later on.

Number is certainly different from many other properties of things. We have seen that number-predications require plural subjects (for numbers greater than one) if they are to be true. Further, number-properties do not distribute to the members of manifolds, whereas the sorts of predicate Frege had in mind as paradigmatic 'physical' property-predicates, e.g. 'green', typically are perfectly distributive. My theory recognizes that number-properties cannot be simply identified with the way an individual can be split into parts. In this, the theory is more sophisticated than the somewhat naive one adopted by Armstrong. The idea that things literally have to be brought together into one place in order to be enumerable receives no credence at all on the present theory. Finally, we can account as well as Frege for the universality of number, without his dubious doctrine of concepts. This universality is not dependent upon a prior decision as to what there is, as troubles Armstrong's account. For, even more than the relations of part and whole,

the relations of identity and difference are ubiquitous and 'topic-neutral'.

§ 5 Counting and the Mereological Properties of Aggregates

There is, however, a group of interesting considerations which cluster around what we might call *homogenous* manifolds, those which can without strain be thought of as composing an aggregate²⁶. I am interested in the conditions under which such manifolds exist, i. e. when an aggregate, together with a suitable count noun, yields a manifold by division. This will then provide a partial justification for including a paper on the philosophy of number in a collection devoted principally to questions of mereology.

Let us call an individual an *aggregate of cs* when it is composed of a whole number of *cs* and nothing else besides. More explicitly, *A* is an aggregate of *cs* iff every part of *A* overlaps some *c*, and every *c* which overlaps *A* is part of *A*. Now on the naivest view of how aggregates of *cs* fall into kinds according to the way in which they divide into manifolds of different numbers, we could divide aggregates of *cs* by the following definitions:

A is a one-aggregate of *cs* iff *A* is a *c*

A is a two-aggregate of *cs* iff *A* is the disjoint sum of a *c* and a one-aggregate of *cs*

A is a three-aggregate of *cs* iff *A* is the disjoint sum of a *c* and a two-aggregate of *cs*.

In general

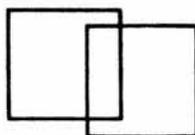
A is an $(n + 1)$ -aggregate of *cs* iff *A* is the disjoint sum of a *c* and an *n*-aggregate of *cs*.²⁷

This recursive specification of these various kinds of aggregate of *cs* only works for those finitely divisible into *cs*. We may cover all sizes of infinite aggregates by the stipulation that an aggregate of *cs* is infinite iff it contains as proper parts aggregates of *cs* of all finite adicities. It then turns out that an aggregate of *cs* is infinite iff it is the disjoint sum of a *c* and an infinite aggregate of *cs*.

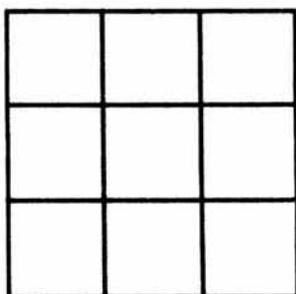
The reason this naive account of when we can divide an aggregate countably into parts which comprise a manifold is not sufficient may be

seen from the previous figure. This is an aggregate of squares, and it does not fall into any of the kinds given by the recursive definition, which is just as well, since it is an aggregate equally well of three, four (two ways) or five squares. The account must be broadened to explain how we may count such squares even when they overlap.

Suppose different *cs* have no common part, are disjoint. Then *cs* will be said to be *absolutely discrete*. *cs* which are such that no common part of two *cs* is ever a *c* will be called *relatively discrete*. If *cs* have no proper parts, they will be called *absolutely atomic*: if no proper part of a *c* is ever a *c*, *cs* will be called *relatively atomic*. Absolute discreteness and absolute atomicity entail respectively relative discreteness and relative atomicity. Relative discreteness and relative atomicity come to the same thing: one *c* is a proper part of another iff the two different *cs* overlap to the extent of the smaller. On the other hand absolute discreteness and absolute atomicity are different properties, the latter entailing the former, but not vice versa. Apples, to take our previous example, may be counted as absolutely discrete, but they would hardly provide much nourishment if they were absolutely atomic. Relative atomicity (or discreteness) does not entail absolute discreteness, since in the figure given below no proper part of a square in the figure is a square in the figure, but the two squares overlap.

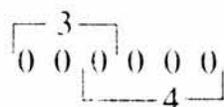


Consider the following linear figure. How many distinct subfigures

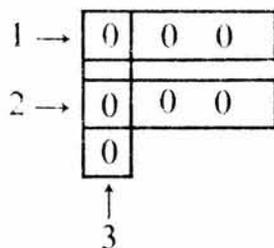


does it have which are squares? The answer is fourteen. These square subfigures are not all discrete from one another: some share vertices or

edges. The square figures are in fact relatively atomic. Were we to consider instead the filled-in squares, i.e. the square figures together with their interiors, we should lose this property, but we still get the answer fourteen, so I shall for brevity consider only the linear case. We can therefore settle the number belonging to the manifold of square figures despite the aggregate of these figures not being, in the sense given by the previous recursive definition, an aggregate of disjoint figures. Yet disjoint collections are clearly important in the philosophy of number, and this importance carries over to the philosophy of arithmetic. We do not illustrate the sum of four and three in the following way:



nor do we illustrate the product of three with itself in this fashion:



but rather take the sum to be of two disjoint manifolds, the product also a certain number of disjoint manifolds all of the same number. I venture to suggest that we should be unable to handle overlapping manifolds like the squares unless we had first built up the ability to recognise manifolds of different number in the disjoint case.²⁸

We may split the linear figure into discrete elements consisting of 16 vertices and 24 open sides (i.e. edges between vertices, lacking their endpoints). Were we dealing with the filled-in squares we could add the nine open interiors. Now various relations of contiguity hold between vertices and open sides, and relations of being parallel and being perpendicular hold among the open sides. Were we to give each of the 40 elements a name these relations could be spelled out in a finite number of sentences, but I shall not go to this length.

We now define a *line* as the mereological fusion of an open side with its two contiguous vertices. Lines, unlike the elements, are not disjoint,

overlapping in vertices. A *linear figure* is then any mereological fusion of lines as defined. The whole figure itself is a linear figure.

A linear figure has a *free* vertex iff the vertex is in (i. e. is part of) the linear figure, and is part of only one line in the figure.

A linear figure has a *junction* vertex iff the vertex is in the figure and is part of more than two lines of the linear figure.

Vertices in a linear figure which are part of exactly two lines are called *corner* or *straight* vertices, according as the lines they are part of are perpendicular or parallel respectively.

A *square* is then a linear figure containing no free vertices, no junction vertices and four corner vertices, such that between any two adjacent corner vertices there are the same number of straight vertices. Squares are thus uniquely determined by their corners.

I contend that we can count or calculate the number of squares in the figure because we can break it up into disjoint elements in this way, and determine which aggregates of these elements are squares. It might be objected that as a general account of when an aggregate can be divided to yield manifolds this is circular, since it depends on the ability to settle the number of certain manifolds in the figure, such as the four corner vertices of a square etc. The objection is misplaced however, since what we are using here is not the general account but the more limited one already given for settling the adicity of manifolds whose members are absolutely discrete. Here we are determining numbers of open sides and vertices, which are absolutely discrete, by construction. This is why they were called *elements*. There is no need for the elements to be all alike, as this example shows. We here have both vertices and open sides as elements.

We now define relative discreteness in a new sense. A manifold of *cs* will be said to be *discrete relative to its ds* iff:

- (1) every *c* is an aggregate of *ds*
- (2) the *ds* are absolutely discrete
- (3) two *cs* are different iff they contain a different manifold of *ds*.

Where *ds* are absolutely discrete they may be counted or calculated by the simplistic way given before. Thus, in our example, linear figures in general, and squares in particular, are discrete relative to their elements (vertices and open sides). When we actually decide for ourselves how many square subfigures there are in the figure, we do not need to go through the rigmarole of dividing the figure into elements and determin-

ing the conditions for when we have the same square and when we have a different one. But it is my suggestion that the much more intuitive method of 'seeing' square *Gestalten* in the figure has its basis in the fact that such a rigorous method is possible. My suggestion, for what it is worth, is that we can only divide an aggregate into a manifold in a certain way if the manifold is of a kind of thing which is discrete relative to some constituent elements. Of course if *cs* are absolutely discrete, they are thereby relatively discrete, being their own elements. This condition of relative discreteness goes some way to clarifying a remark of Frege's:

Only a concept which marks off what falls under it in a determinate manner, *and which does not permit arbitrary division of it into parts*, can be a unit with respect to a finite number.²⁹

The unwanted arbitrary divisibility is excluded here precisely by the condition that whatever falls under the concept *c* must be an aggregate of elements which are absolutely discrete. It would not matter for this requirement's being met if there were infinitely many such elements in a *c* so long as there were some effective way of deciding which *c* is which.

It seems to me that this requirement of relative discreteness (in the second sense) shows that there is a much closer connection between mereological considerations and those pertaining to what kinds of common nouns are sortals. Griffin, for instance, explicitly rejects all mereological criteria for sortality, in favour of one connected with countability:

A term 'A' is a sortal iff there *can* be cases in which 'A' provides, without further conceptual decision and without borrowing other principles of individuation, principles adequate for counting As.³⁰

However, since it is a requirement for being able to count, or, at least, for a common noun as applied to an aggregate to yield a manifold with a definite number, that the noun satisfy the mereological condition of relative discreteness, countability itself in such cases presupposes mereological principles. Indeed it would seem that the only region in which sortal terms do not effect a division of reference is one where we should have doubts about there being any antecedently given 'material' upon which the sortal terms work anyway, namely the abstract realm of mathematical objects. It is decidedly odd to think that numbers, for instance, are given to the mind firstly as an 'amorphous lump' or undivided aggregate, which the mind, with the use of the reference-dividing sortal noun

'number', then proceeds to slice up into individuals.³¹ The idea of an aggregate of numbers, i.e. an individual which contains numbers as its (perhaps atomic) *parts*, is a strange one. There may indeed be ways of making it work. But I cannot escape the conviction that this is a *transference* of the idea of whole, part and aggregate, to a region where it is not ultimately at home: that numbers, in so far as they are given to us at all, are given *already* as individuals. We could only gain access to an aggregate consisting of numbers *via* the sortal noun 'number', i.e. *via* the individuals comprising the aggregate. Hence it seems to me that where sortals serve genuinely to divide reference, to articulate for us the world into individuals, mereological criteria for sortality apply.

The first part of Frege's passage quoted above speaks of a concept's marking off what falls under it in a determinate manner. This yields a further condition which a sortal noun (phrase) must satisfy if it is to be able in a given context to determine a manifold: it must not be vague. Vagueness is not necessarily determined simply by the noun alone. A noun or noun phrase may easily determine a manifold in one context and not in another. For instance, while watching a standing wave experiment in a hydrodynamics laboratory I may observe 'These waves have a clearly sinusoidal shape', and the plural expression 'these waves' may pick out a precise number of waves. But if, say, I am standing on the deck of a ship watching a heavy sea, and I observe 'These waves are over twenty feet high', then while what I say may be perfectly true, it is very unlikely that there could be any answer to the question 'How many waves were referred to by the expression "these waves"?' The same is more obviously true of an expression like 'the waves now in the Atlantic Ocean', which has at least three areas of vagueness attached. Firstly there is the problem of just how far the Ocean extends, so we know which waves are in it and which are not. Then there is the problem of which particular watery disturbances are to count as waves and which are just regions of general turbulence, etc. Finally there is the problem that there is no general recipe for deciding where one wave ends and another begins. This arises where wavefronts converge, catch one another up, etc. For all that, we may make true predications about the waves in the Atlantic Ocean: We may say, e.g., on an abnormally stormy day, that the waves in the Atlantic Ocean are overall larger than they usually are.

The problem of whether a noun brings with its meaning a general recipe for dividing reference, settling the boundaries of the individuals falling under it, has been treated at length in recent philosophy under

the heading of 'criteria of identity'. We shall not here dwell long on this topic. Suffice it to say that the wave example shows that it is not a necessary condition of a sortal noun's being able to determine a manifold that it should possess an associated criterion of identity which applies in *all* circumstances. We require only that it should present one with such a criterion which may work in *some* circumstances.³²

Mass nouns may sit within terms which do not designate individuals or manifolds, and they may also sit within terms which do. For instance, the plural term 'the slices of beef on my plate' may determine a manifold, but not because the noun 'beef' does anything towards this: rather the noun phrase 'slice of beef' divides reference here. By contrast, expressions such as 'the beef on my plate' or 'the gold in this ring' do not designate individuals or manifolds, except in so far as each expression determines, or, as I shall say, *delineates*, a certain individual which is the aggregate of all the beef on my plate or all the gold in this ring. Indeed this is the way in which Quine accounts for the use of mass nouns in subject position, as e.g. in 'Snow is white': the term 'snow' here denotes, according to him, the single scattered individual which is the heap or aggregate of all snow.³³ Whether this proposal is right or not I cannot tell: there appear to be few considerations urging either an acceptance or a rejection at present. However, not all uses of terms containing mass nouns are to be so explained: in particular those which already parcel the stuff up in some way, e.g. 'the ingots of gold in Fort Knox'.³⁴ The ability to make true predications using terms containing mass nouns, even without it being necessarily the case that such a use needs to designate an individual (as distinct from delineating one) is closely parallel to the ability to use plural count nouns in a similar fashion, as mentioned above. For on such occasions we can regard the plural count noun as performing a role just like that of a mass noun: it is applicable to a more or less vaguely delineated sector of reality, but does not determine a manifold. We could imagine a suitable mass noun being used with equal facility, e.g. rather than talking about waves more than twenty feet high, one could talk of a twenty-foot *sea*. Here the question 'How many seas . . .?' is not only without answer: it is patently the wrong one. The noun 'sea' is here being used as a *mass* noun, albeit that it can also be used also as a count noun. The question is as ill-formed as the question 'How many golds are there in this ring?' The 'How many . . .?' question simply has no application in connection with a noun used as a mass noun. The same consideration therefore applies to those uses of plural

count nouns where a mass noun would be equally appropriate, even though the question 'How many . . .?' is not here *grammatically* wrong.³⁵

I spoke of a term incorporating a mass noun as *delineating* an individual (whether or not it also *denoted* it). The same can be applied to those plural terms which designate a manifold which can be heaped to form an aggregate. We may describe the aggregate as that one *delineated* by the plural term. Two plural terms designating different manifolds may yet delineate one and the same aggregate, as we know from various examples. In such a case we may say that the manifolds *coincide*. The aggregate as such could also be delineated by a term using a mass noun. For example 'the aggregate of the slices of beef on my plate' could delineate the same aggregate as 'the aggregate of beef on my plate' (provided all the beef on my plate is in slices). This provides another reason for thinking that sortal nouns applying to abstract regions do not effect a division of reference, since any mass noun applying in such a realm would surely have to be defined *via* the sortal terms, whereas in other, more concrete regions, the sortal noun effects a division of reference on an aggregate to which we may have access other than as the aggregate obtained by heaping together as an individual the members of a manifold. In the development of cognition, division seems to be more primitive than composition, except in the atomistic psychologies of the empiricists. In examples drawn from everyday life, however, both division and composition may play a part in determining a manifold designated by a plural term. For example, a settler arriving in virgin territory determines his land by dividing it from the rest, and he may then determine the estates of his various sons by further dividing the land which is his. But he may also determine these estates by composing or putting together plots which have already been divided, such as fields or strips. This example suggests that the manifold, which is, as it were, something which has both *internal* and *external* boundaries (external ones which divide the aggregate so delineated from the rest of the world, and internal ones which divide the aggregate up into the members of the manifold) is no more and no less a product of human thought and action than the aggregate itself. If we regard the aggregate as already existing before the settler arrived, and think that he merely picked it out from among all the other aggregates of land there, there is no reason to suppose that more boundary-fixing will bring something new into existence when the first episode of boundary-fixing did not. If on the other hand one wishes to

say that the settler brought the various plots into existence when he fixed their boundaries, which are internal to those of the original plot, then one must say the same about the original plot itself. To suppose that while the sub-plots did not exist before the act of demarcation while the large original plot did so exist is to manifest prejudice in favour of the singular again: manifolds are no more *and no less* objective and mind-independent than individuals.

Spatio-temporal coincidence has often been held to be a sufficient criterion of identity for material objects. This may go some way towards explaining the temptation to identify a manifold, especially a material manifold, with the aggregate delineated by the plural term designating the manifold, and hence to overlook manifolds and plural reference. Indeed many such manifolds share many of the properties of their associated aggregates. A manifold of things which are red may itself be regarded as red, just as its aggregate is red. A manifold is not to be regarded as identical with a complex individual, however. A complex individual is one thing with many parts. It is the aggregate, not the manifold, of its parts. Of course, it is not *merely* the aggregate of them, since its parts are in various determinate relations to one another. A manifold, on the other hand, has members rather than parts. It is many individuals rather than one. It is wrong to suppose that 'being' and 'being one' mean the same thing, as Plato knew.³⁶ This fact has however got lost with the tendency to concentrate exclusively upon the singular. One good reason for refusing to identify an aggregate with its coincident manifolds is that we should then identify the manifolds, and lose the desired mutual exclusivity of number-predicates. Another good reason is that the aggregate and a manifold which coincides with it may exist at different times. For instance, a certain aggregate of clay may exist long before it is fashioned into a number of clay vessels, and may still exist after they have been broken up. Conversely where individuals may survive despite part-replacement, as organisms may do, an individual may exist long after the matter which made it up has become widely dispersed, perhaps even partly annihilated.

§ 6 One and Zero

There is still the question as to what we can do with the numbers 0 and 1 on the present theory. The number one is easy to accommodate. We may

regard an individual as a degenerate case of a manifold. It is not many, but one. However, we need not discriminate against it on those grounds. As mentioned above, it is not an empty predication to say of something that it is one thing, since there are true predications with plural subjects. Indeed in certain cases a plural expression may succeed in designating only one thing. For instance, someone with a shaky grasp of Roman history may believe that Livia married both Octavian and Augustus. Here the expression 'Octavian and Augustus', which looks plural, turns out to be 'logically', i.e. semantically, singular, in much the same way as an expression which is grammatically singular, e.g. 'Queen', may designate a plurality (in this case a rock band). Any individual is as such one thing, just as any pair is as such two things.³⁷

The number 0 presents a greater difficulty. It was Frege's boast that his own theory of numbers as (in the first stage of the theory) properties of concepts was the only one which could account smoothly for the fact that there is a number nought. For a concept may have no things falling under it as easily as it may have some things falling under it. But in the case of a manifold of no things – there is no such manifold, so there is nothing to which we can ascribe the number zero.

Firstly we must ask whether it is such a grievous defect if a theory of number makes a difference between zero and all the other 'positive' numbers. We all recognize the difference between a positive answer and a negative one to a question such as 'Are there any biscuits left in that tin?' The answer 'There are none', while not itself to be regarded as a rejection of the question (as e.g. we should have to make if there were no tin there referred to) is a very different answer from 'Yes, there are just three left'. We all know that the sign for zero, and the concept that goes with it, are later inventions than signs for the other numbers, even on the advanced Indian-Arabic numbering system. The number nought is and was always felt to be something of an invention by comparison with the 'positive' whole numbers, more on a par with negative numbers than with the positive integers. Why is it a defect of a philosophy of number that it should explain this difference? Of course Frege too had an explanation of the difference, but this explanation makes rather too light of the conceptual jump required to arrive at the number 0. In our view it is precisely because there is no manifold or individual denoted or designated by an empty term that we must look for a different explanation of the number 0 from the others. Of course a concept may have no objects falling under it, but cannot have – 3 objects falling under it. But we have

an explanation of that also. We may accept that it is true that there are no *cs* without being forced to Frege's conclusion that a philosophy of number is inadequate unless this is taken into account. For if there are no *cs*, the term 'the *cs*' is empty, i.e. designates nothing, i.e. *does not designate*. Therein lies the difference between zero-predications and positive predications of number. We do not need to conjure a null manifold or object into being to bear this property: with a robust sense of reality we maintain that there is nothing that is nothing. That is not to say that we have no explanation as to what is going on when we deny that there are any *cs*: we are simply doing just that. Only by a later convention do we take that to be a *numerical* predication. In Chapter VIII of *Philosophie der Arithmetik* Husserl draws an illuminating analogy which enables him, so it seems to me, to win this point hands down against Frege. If we accept that numbers correspond to answers to "How many?" questions, even the negative answer "None", we could similarly treat places and times as corresponding to answers to "Where?" and "When?" questions respectively. But if we follow Frege's argument, we should by parity of reasoning have to accept that there is a special place called "Nowhere" and a special time called "Never". The absurdity of this position ought to reflect back on Frege's position on the number zero. If it is felt that the case is here weaker, I would suggest two reasons: firstly the avowed *utility* of zero in arithmetic, and secondly the dulling of the senses brought about by generations of passive acceptance that Frege was right.

If we wish to establish a convention about what a null manifold is, we can only do so within the scope of a theory of manifolds as determined by sortal terms. It will not work for a general theory. In certain circumstances we can say that individuals or manifolds are null-manifolds. An apple, for instance, or several sheep, are all null-manifolds-of-insects. That is to say, they are none of them insects. But it seems unwarrantedly artificial to take such a step simply to have the satisfaction of providing a bearer for a special null property. In any case, *being neither an insect nor many insects* seems to be a predicate corresponding to no property.

§ 7 There are no Second-Order Manifolds

There appears to me to be just one obstacle to be removed before we can accept the theory of numbers as properties of manifolds (and degen-

erately, in the case of 1, individuals). It is not one which we find among Frege's list as such, though it arises from a special case of considering the ubiquity of number. The problem is that we can count, not merely individuals, but also manifolds of individuals. Suppose that in looking along a wall I see four pairs of chairs arranged as shown. Then I may count the pairs just as easily as I may count



the individual chairs. Indeed, because of the peculiar arrangement of the chairs I may even count pairs of pairs. This makes it look as though I am counting manifolds of manifolds.

At first sight it looks as though there is an easy way out of this. It is to suppose that in indulging in such counts we are in fact counting aggregates which consist of chairs, rather than manifolds. We count first the aggregates which have as parts two chairs close together, and then we count the aggregates which consist of two of these two-chair aggregates close together. There is indeed some plausibility in the suggestion in this particular case, the more so as both the pair-aggregates and paired-pair-aggregates are here absolutely discrete. The suggestion amounts to taking the bearers of the number-properties so ascribed to be not manifolds of chairs but aggregates of chairs of a certain kind, i.e. certain complex individuals with chairs as parts. While the suggestion appears both harmless and profitable in the present case, we cannot apply it in all cases because not all manifolds have the nice discreteness and atomicity properties we have here. Take the case of the three overlapping squares with two smaller squares at the overlaps, considered before. We may remember that there were precisely four different manifolds of squares all delineating the same aggregate, one of three squares, two of four and one of five. But precisely because they do all correspond to the same aggregate we cannot say that we count these manifolds by counting the associated aggregates, since there is only one such, whereas the number of manifolds is four.

It is worth having recourse here to the concept of relative discreteness mentioned before. We may count *cs* even though they are not absolutely discrete, so long as they are aggregates of *ds*, which are. Now we might apply parity of consideration to manifolds. A manifold is determined by its individuals. "Two" manifolds are the same which have the same

members, i.e. comprise the same individuals. It is for this reason that we may have such things as true plural identity predications. This means that the identity of a manifold is determined by the several identities of its members. So long as we can discern the individuals of a manifold we can tell whether or not it is identical with a given manifold. Two manifolds may be distinct but overlap, because they have some but not all members in common. One may indeed be a submanifold of the other. It is important to recognize the difference between the sense in which individuals may be said to overlap, namely when they have a common part, and that in which manifolds may be said to overlap, namely when they have a common member. (There are connections between the two notions of course.) If we allow individuals, i.e. members, of manifolds as submanifolds, then every manifold of n members has $2^n - 1$ submanifolds, including the manifold as a submanifold of itself. The difference between this and the number of subsets of a Cantorian set (2^n) lies in the fact that there is no empty manifold. It is because manifolds are determined by their members that we can count them. Once again, the more complex ability, that of counting manifolds, rests on and presupposes the ability to count individuals, just as the ability to count overlapping individuals depended on the ability to count discrete ones. The difference between the chairs and the squares lies in the fact that the squares in the figure were aggregates of their elements, while the pairs are manifolds of their individuals.

But now two questions arise. If we can thus count manifolds, are not the manifolds new unities in their own right, distinct from the several individuals which make them up, contrary to what I have maintained? We can after all speak of four pairs of chairs, two pairs of pairs of chairs etc. The expression 'this pair of chairs' is singular, as is 'this pair of pairs of chairs'. Here I wish to maintain the deflationary position that manifolds are not new, higher-order, unities distinct from their several members. Of course, each manifold is distinct from each of its members; provided it has more than one, that is. But that tells us nothing new. The ability to count manifolds I have already explained. The fact that we have a singular expression 'pair of chairs' may be taken as a mere slip of syntax. We do after all need to know how many individuals make up a pair in order to be able to understand the word. Of course, some senses of 'pair' mean more than just 'two of a kind'. They mean two of a kind which are matched by a certain relation, as e.g. a pair of shoes are matched in style, size, colour, and by being one left and one right etc. The same remarks

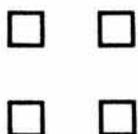
go for e.g. pairs of gloves, duelling pistols etc., and *mutatis mutandis* for other sets of things which are meant to 'go together' in a certain way. Such sets are likely to be discrete from one another, so there is no difficulty in accounting for our ability to count them in either of the two ways mentioned above. Languages are well-endowed furthermore with *collective* nouns which go in the singular but which, when attached to count nouns in the plural, serve frequently to help in designating manifolds. The fact that the term itself, e.g. 'this collection of stamps', 'this group of people' is in the singular does not divert us from recognizing that the expression designates a plurality of individuals. Of course what makes us say certain pluralities are a collection, a group, a set etc. is frequently much more than their being simply several of a kind. Not just any plurality of stamps constitutes a *collection*. For that, common ownership is required. It is frequently appropriate to regard such multiplicities as constituting higher-order objects in their own right. A stamp collection may grow and still remain the same collection, but contain a different lot of stamps. We might say that it is constituted or made up by different manifolds of stamps at different times. It cannot be identical with any such manifold, since they, but not it, are determined by membership. All the same, an expression like 'Uncle Harry's collection of stamps' may at different times designate different manifolds of stamps, *per accidens*, through denoting a collection with a certain membership, namely those stamps making up that manifold.

There are several ways in which manifolds get counted, not just one. This fact has become obscured by the blanket tendency to treat all pluralities or higher-order entities alike as sets, which are understood in an abstract, Cantorian fashion, as individuals distinct from their members, and distinct even from their members taken together. Such entities constitute Russell's 'sets-as-one', as distinct from our manifolds, which are his 'sets-as-many'.³⁸ Some counts which we might construe as counting manifolds are in fact counts of institutional and higher-order objects, which may indeed be regarded as individuals. For instance we might count the orchestras playing in a certain city. There might be, say, three such orchestras, even though one of them contains just the same players as another. One and the same manifold of players gets counted twice over, without error, in this case, because orchestras, unlike manifolds, are not determined by their members. Other counts of manifolds do treat them just as many individuals rather than as higher-order objects, after the fashion described above.

This raises a serious problem. Is there a different sense in which several individuals are, say, three, and several manifolds are three? We have to answer both yes and no. The answer is negative in so far as the concept of identity is applicable to manifolds as well as to individuals, without that making manifolds new individuals. If we allow a neutral identity predicate, which applies as well between plural terms as between singulars, then this same predicate applies to predications of the form 'A and B are two', i.e. 'A and B are different', and so on through the other forms of number-predication. The answer is positive in so far as we *may* distinguish between a singular identity predicate and a plural or neutral identity predicate. If we count three manifolds, then that manifold which is their union is three in a different sense from that in which it is however many individuals it is. That sense in which a manifold is many individuals must be regarded as the primary sense, and the sense in which a manifold may be many when we are considering submanifolds of it, as secondary. This implies that we are rejecting the concept of a second-level manifold, that is, a manifold of manifolds which is distinct from any manifold of individuals. Just because we can identify and count manifolds does not mean that we can make manifolds of them. Of course, given several manifolds, we may consider that manifold which is their union, i.e. the manifold comprising just those individuals which are members of any of the several manifolds. But this is once again a manifold of individuals. The several manifolds from which it is formed are not its members, unless these several manifolds already happen to be individuals, i.e. unit manifolds. For a manifold is just the one or many individuals designated by a term. When we consider a plural term with other plural terms as subterms, e.g. 'the Smiths and the Browns' then the individuals designated by this term are just those which are designated by at least one of the conjoined plural subterms. If on the other hand we count *families*, then, in so far as we may consider families as higher-order entities, we can count here just the two. (The problem with families is settling a criterion of identity which will account for the possible intricacies of intermarriage, but in this case it might be that no question of close relation arises.)

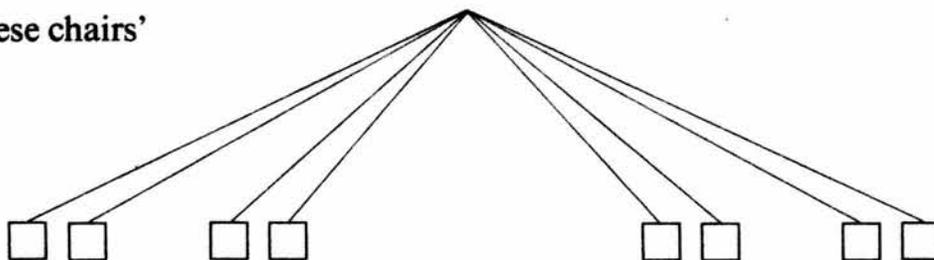
It has frequently been maintained that mathematics would be crippled if we did not have classes of classes, classes of classes of classes etc. While much of this dependence has come about through the reinterpretation of a great many mathematical entities as sets of various other entities, including also sets, it does look as though there are areas where sec-

ond and third-level classes are called for, e.g. in combinatory problems like 'How many ways can I pair socks in a drawer containing twenty-four socks of which twelve are blue and twelve are brown?' It seems to me that the apparent requirement for manifolds of manifolds, in order to make sense of questions like this, has been overrated. We grant that there is a type distinction between individuals and the manifolds of which they are members. So the sense in *which the four manifolds of overlapping squares form the same aggregate is a different sense from that in which these four squares*

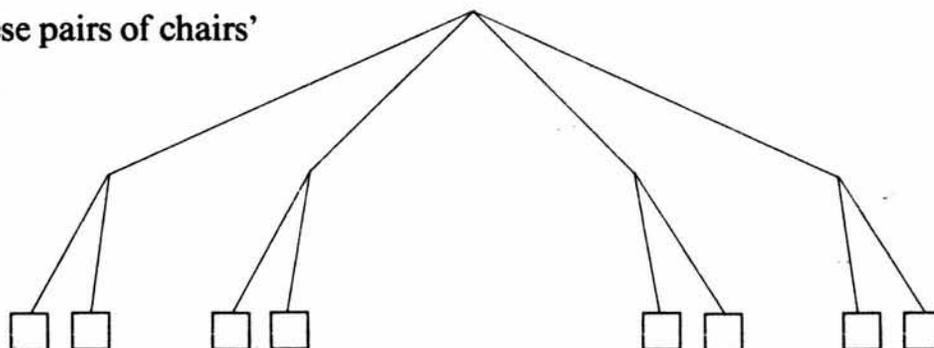


form a manifold. It is usually supposed without question that this is sufficient to motivate an infinite hierarchy of types. This I believe is not so. There is scant need if any to ever consider *manifolds* of manifolds, though we may often wish to consider together a *number* of manifolds. Of course we may regard our considerings-together as showing up the various possibilities. We might look upon different plural expressions as relating to the objects in a different way. Consider the case of the chairs against the wall. It might be that we should wish to look upon the role of different referring expressions like this:

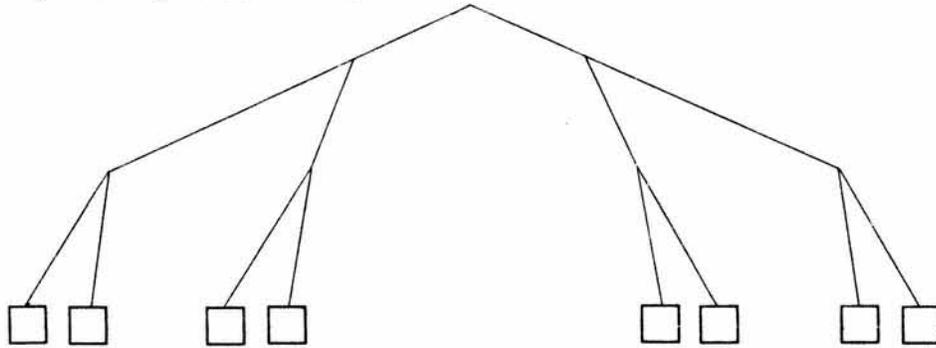
'these chairs'



'these pairs of chairs'



'these pairs of pairs of chairs'



It is one thing, however, to draw diagrams like this showing how we may group and subgroup individuals into larger or smaller groups: it is quite another to think that we have made any semantic sense of these diagrams in terms of higher-order manifolds. The individuals designated by a term are represented by squares. We have represented subterms by nodes in the tree. To take there to be manifolds as entities distinct from the many individuals comprising them is to treat higher nodes just as if they were nodes immediately above the individuals. But the diagrams here carry a little plausibility because they begin to treat nodes or their associated manifolds just as new individuals alongside, but slightly different from, the original ones. We can of course treat nodes, expressions, or any other individuals for that matter, as *representatives* of manifolds. In this way axiomatic set theory may be regarded as a theory of individuals (sets-as-one) representing manifolds (sets-as-many).³⁹ There is then no problem about allowing that there are sets of representatives of sets, which are equivalent to sets of sets in the usual interpretation. In the diagrams, a node is treated as though it were an individual, whereas the semantics of the situation tells us that it is not. If we wish to represent e.g. a pair of individuals, the best way is not to draw a pair connected to a node, but simply to draw a pair of dots. These *are* a pair. When we refer to the manifold consisting of A and B on the one hand, together with B and C on the other, where A, B and C are all individuals, then we have referred to simply the manifold of A, B and C, albeit in a somewhat redundant way.

If we look at combinatorial problems which are usually interpreted as involving counting classes of classes etc., we find that what is actually asked for in the count is often something like a number of possible arrangements, orders, selections, combinations, partitions and the like.

We may easily count or calculate such numbers without appeal to higher-order classes, which are introduced only for the sake of uniformity. The counting or calculating procedure very often takes the form of *proxy-counting*, that is, letting some order, arrangement or whatever be represented by something else, such as a sequence of symbols, and counting or calculating the different orders or kinds of such sequences, modulo some equivalence relation such as equiformedness in many cases.

While this is no more than a sketch of the intricacies involved in counts which proceed by proxy, it is of course clear that there must be some general rule enabling us to get from the proxy objects to their originals. This need not be a simple one-one correspondence. By setting up such rules we can enumerate manifolds of entities which could not be presented to us one by one for a more basic kind of count, where we tell off the objects in turn as they are presented. We are thus enabled to settle the number of manifolds of entities which cannot be given to the senses, or which cannot all be presented together, such as populations of a country, or ways of drawing two million pairs of socks out of a drawer . . . , or mutually exclusive possible events or states of affairs.

In this way we hope to have shown the plausibility of the idea that individuals and pluralities belong together in the lowest ontological type, albeit not a type in the usual sense consisting only of individuals, but lowest in Frege's sense of being objects rather than properties or concepts. In a sense, it is wrong even to speak of 'individuals and pluralities' as though, like cats and dogs, they could exist without one another. Certainly a plurality is never *identical* with any individual, but they are ontologically inseparable: whoever admits the existence of at least two individuals admits that of at least one plurality, even if, like the man who had talked prose all his life, he had never realised it. While one can, it seems to me, consistently affirm individuals and deny sets (indeed some, e.g. Goodman, have done just this), one cannot likewise affirm at least two individuals but deny pluralities, for the plurality of two objects just is *them*. It would be like affirming that something is red and also affirming that it is round, while denying that it is both red and round. The penalty in each case is the same: formal contradiction.

Thus predications of number attribute properties to manifolds, in the basic sense when we consider the number of individuals, in a derived sense when we consider a number of manifolds which are not all singletons, i.e. individuals. The analogy between the two senses is provided by

the fact that we can express both using an identity predicate (whether absolute or sortal-relative) which is neutral as between singular and plural identity. In very many cases where we may consider the individuals in a manifold as heapable together in an aggregate, the possibility of counting or calculating how many such individuals the said aggregate divides into is secured by the individuals' falling under some sortal noun which is discrete relative to some other noun giving elements of these individuals. Where mathematics has erected higher-order manifolds or sets, this is an artificial construction which has been introduced to unify the treatment of various branches of the subject. Sometimes, as for instance in geometry, a set-theoretic treatment can be replaced by a mereological one without detriment, and with added intuitive content. I hope therefore to have shown that Frege's objections to making number a property of 'external things' can be met. Clearly however before we can say that a rounded philosophy of number has been provided, more work needs to be done on the various conditions which enable terms to designate manifolds upon occasions of their use, and on the various ways in which we are able to settle the number of such manifolds. The essential task before us is to build a bridge which will connect, by means of an adequate theory of number, the philosophy of number with the philosophy of arithmetic.

Notes

References in these footnotes are to the works listed in the bibliography at the end of these essays. Works are cited under the name and year in which they are listed there.

For this essay in particular the following abbreviations are used:

Pr Russell, Principles of Mathematics (Russell, 1903)

Grl Frege, Grundlagen der Arithmetik (Frege, 1884) (Page references are to the 1953 German/English edition.)

¹ See Benacerraf, 1965.

² I should here make some remarks on the word 'manifold' itself. It is sufficiently uncommon in modern usage for me to be able to annex it for my own purposes. The word was more common in nineteenth century books, where it was roughly synonymous with 'set', 'class', 'aggregate' or 'collection'. Cantor for instance used the German equivalent *Mannigfaltigkeit* before later changing to *Menge*: cf. Cantor, 1932. I prefer to reserve the word 'aggregate' for mereological sums or heaps. I have avoided the terms 'set' and 'class' because they have too close a connection to the tradition stemming from Peano and Frege which I oppose. For more detailed discussion cf. the third essay below. The

word *Mannigfaltigkeit* as used by Husserl is translated variously as 'manifold', e.g. by Findlay, and 'multiplicity', by Cairns. Given Husserl's avowed intention to broaden the original mathematical concept of Riemann (cf. Husserl, 1929, Ch. 3) and the practice of using the word 'manifold' in English-language mathematics for the Riemannian concept, it might seem that this is the better translation in Husserl. However Husserl's concept is far wider than the mathematical one, and most closely approximates the modern concept of *model*, in which sense the term 'manifold' is used by Null and Simons in their essay in this volume. For this reason I prefer Cairns' translation. This still allows the unequivocal alternative 'plurality' (*Vielheit, Mehrheit*) for my concept. The relation between my concept and Husserl's is this: both are generalisations of the idea of an object or referent of a mental act or linguistic expression. Whereas Husserl's *Mannigfaltigkeiten* are the referents of formal theories, mine are the referents of plural referring expressions.

³ *Grl* §§22–4.

⁴ When I mention common nouns I shall also mean common noun phrases. A common noun phrase of the form c (c which Φ s) is + count iff it is grammatically congruous to say something such as 'there are three c s (which Φ) in Mongolia'. Cf. Griffin, 1977, 23. I shall refrain from using the word 'noun' for proper names, which are among what I call *terms*.

⁵ Frege, in a letter to Husserl of 24. 5. 1891. Frege, 1976b, 96–7.

⁶ *Grl* § 46, p. 60.

⁷ See e.g. Quine, 1960, 90f.

⁸ For different treatments of quantifier phrases see my 1978 and Evans 1977. My use of the word 'term' differs sharply from that of Russell in *Pr*. For me a term is always an expression, whereas for Russell it is the object or referent of an expression.

⁹ See e.g. Geach, 1962, 31 f.

¹⁰ *Pr* § 70, p. 69, n., where it is asserted that propositions with a plural subject have not one, but many logical subjects. Cf. further the third essay below.

¹¹ *Pr* § 61, p. 59, where Russell takes 'all a s' to denote a_1 and . . . and a_n , whereas 'every a ' denotes a_1 and . . . and denotes a_n (n always finite). There is surely here a distinction without a difference. What perhaps leads Russell to over-subtlety is the grammatical difference that the first is singular while the second is plural. Since the distinction which I make between singular and plural reference does not pertain at all to such quantifier phrases I need not even consider Russell's position. It is part of the side of *Pr* that I find unacceptable that Russell did not distinguish reference from quantification. But there is more to Russell's extension of 'denote' than its illegitimate use with regard to quantifiers.

¹² Quine, 1966, reprinted in Klemke, 1970. The passage mentioned is on p. 6 of the reprint.

¹³ In Appendix B of *Pr* Russell is forced, because of type theory, to deny that 'Heine and the French' denotes something of the same type as 'the French'. Russell admits that this is against common sense. We can accommodate common sense here: both designate manifolds.

¹⁴ Frege's concepts are *objective* but are not *objects*: they cannot be named. Cf. Frege, 1952, 45.

¹⁵ In Leonard and Goodman, 1940 aggregates are called *fusions*.

¹⁶ *Pr* § 70, pp. 68–9.

¹⁷ Armstrong 1978, Vol. II, Ch. 18, pp. 71–4.

¹⁸ Sprigge 1970, pp. 112–30. The diagram of overlapping squares is taken from p. 116.

¹⁹ McTaggart 1921–7, Ch. 15. My manifolds differ from McTaggart's groups in that I do not allow what he calls *repeating groups*, such as A, B, A, B. For me this is identical with A, B. Cf. the third essay below. Moreover McTaggart allows that groups may have

- parts, whereas I hold that only where the members of a manifold admit of a sum or aggregate can we talk of parts, and these are not parts of the manifold itself but its sum, except when the manifold is a singleton.
- ²⁰ My manifolds are like Russell's classes as many, Cf. essay 3 below.
- ²¹ There is more than one doctrine of the relativity of identity. Cf. Wiggins 1967, Griffin, 1977.
- ²² It is the weight of such considerations which has dissuaded me of the group view of numbers which I originally held. I am grateful to Barry Smith for so forcing me to stretch my concept of aggregate in order to maintain this position that the concept eventually broke under the strain, and led through to the position of this and the next essay. Another objection to the group view is that it slurs the sense/reference distinction. An aggregate or a manifold is the referent of an expression, but an aggregate *qua* composed thus and so seems to straddle the divide between the thing considered and our manner of considering it. Cf. the remarks in the previous essay on the inducement of opacity by *qua*.
- ²³ Armstrong, 1978, Vol. I, 32.
- ²⁴ Black 1971 discusses in outline the various possibilities for distribution of the predicate.
- ²⁵ The close relation between number and diversity was recognised by Jevons, and by Descartes before him. Cf. the remarks in *Grl*, p. 46. Frege's justified criticisms of Jevons are so justified only as an attack on his theory of abstraction. Jevon's first-stage philosophy of number is preferable in my view to Frege's own, though he did not recognise any difference between philosophy of number and of arithmetic.
- ²⁶ It is of course for homogeneous manifolds that the group theory of number is most attractive. For sparse ontologies, all manifolds are homogeneous. Rejection of the group theory takes Frege's sixth objection to an 'external things' theory more seriously.
- ²⁷ C is the disjoint sum of A and B iff: $C = A + B$ and A is disjoint from B , i.e. A and B have no common part. ($A \sqcap B$ in the notation of Leonard and Goodman, 1940). Here '+' is the mereological, not the arithmetical operator. '+' must also be differentiated from the use of 'and' to form plural terms.
- ²⁸ For similar considerations and diagrams, see Fogelin, 1976, Ch. XV.
- ²⁹ *Grl*, p. 66. (My emphasis.)
- ³⁰ Griffin, 1977, 43.
- ³¹ Cf. the picture sketched by Dummett in his 1973, p. 563 f. Dummett explicitly recognises that relative atomicity is not necessary for countability in a diagram of overlapping rectangles on p. 549.
- ³² It follows that the distinction in Geach, 1962, 39, between *substantival* and *adjectival* general terms is somewhat skew to our purposes. It may be that many sortal nouns possess open texture, so that we are not in possession of an infallible recipe for counting their instances. Cf. Zemach, 1974. But the open texture of sortals does not prevent us from sometimes or often counting under them, though it makes it harder to say in advance of any given noun whether it is relatively atomic, absolutely discrete, etc.
- ³³ Quine 1960, 98.
- ³⁴ Words like 'piece', 'heap', 'ingot' etc. which when used with a mass noun yield a count noun (phrase) have been called 'parcel words'. Cf. Griffin, 1977, 61.
- ³⁵ That plural count nouns behave very much like mass nouns has been forcefully argued by Laycock in his 1972, and is supported in Griffin, 1977, 33; 61. I cannot however agree with Griffin's statement (p. 61 n.) that plural count nouns convey no criterion of identity. This may be true for an example like 'waves', but where there is a good criterion for the noun in the singular its plural inherits the criterion in the plural, so to speak: if I can tell by it when I have the same c twice I can tell also by it when I have the same cs twice.

- ³⁶ Plato actually contrasts unity with identity, but identity and being are related by the formula: to be is to be identical with something. Cf. the following passage from the *Parmenides*:
- PARMENIDES: Well, I think you will admit that the nature of unity is one thing, and that of sameness another.
- ARISTOTELES: Why?
- PARMENIDES: Because a thing does not always become one when it becomes the same as something.
- ARISTOTELES: But why not?
- PARMENIDES: If it becomes the same as the many, it must thereby become many and not one.
- (Plato, 1953, 139d.)
- ³⁷ Even the status of 1 as a number is something of a modern convention. Both Descartes and Hume contrasted unity with number, i.e. plurality. Husserl continues the older usage: cf. p. 12 of his preface to his 1931, where he cites the number series as 2,3,4 . . . This appears to me to be going literally one too far. In *Philosophie der Arithmetik*, p. 129 ff., where Husserl argues his case against 1, he lays far too much stress on the idea that an answer to 'How many?' must be of the form 'so many', where this excludes just one. He is then left in the invidious position of having two *quite different* sorts of negative answer to such questions. The whole tenor of my following paper is both to draw attention to and show how, for scientific purposes, to minimise the awkward grammatical distinction between singular and plural. I would agree here with Frege that 'One' is a perfectly satisfactory positive, one might say singularly positive, answer to a 'How many?' question.
- ³⁸ *Pr* § 70. For Russell a class-as-one is the aggregate of the corresponding class-as-many. Cf. the next essay.
- ³⁹ For a more fully developed account of such representation see § 6 of the next essay.

III. Plural Reference and Set Theory

Most mathematicians do not perceive the problem which is posed by the abstractness of set theory. They prefer to take an aloof attitude and pretend not to be interested in philosophical (as opposed to purely mathematical) questions. In practice this means that they limit themselves to deducing theorems from axioms which were proposed by some authorities . . . the writings of contemporary set theorists and logicians do not offer very much which could help us in solving these problems.

Mostowski, 1966, 140f.

This essay has three aims, only one of which is furthered in detail. The first, and basic one, is to criticise the conventional interpretation of axiomatic set theories as alternatives in a programme of formalising the 'naive' concept of set, collection or class. The polemic which needs to be directed at the various conceptions of set used in defence of this view has already been convincingly accomplished by Max Black and Erik Stenius, so I need not carry that through here.¹ I shall be more concerned with developing a positive account of what I take the naive conception to amount to. The principal idea, which is Black's, is that sets are to plural terms as individuals are to singular terms. In the previous essay I called such entities manifolds. They entered in the context of the philosophy of number, as bearers of number-properties, whereas in this essay I shall consider them for their own sake and in greater detail. Cantor himself was led to abstract set theory through consideration of number, in particular transfinite numbers. It was he who first showed clearly what it means for one infinite collection to have more members than another infinite collection, and showed that there could be collections with different transfinite cardinality.

The positive theory of manifolds will be treated in § 4. §§ 2–3 prepare the way for this. In § 1 I shall suggest that the basic idea of a manifold, or class as many, has a nobler and longer history than Black and Stenius might suggest, and that echoes of this conception still inform some systems of axiomatic set theory.

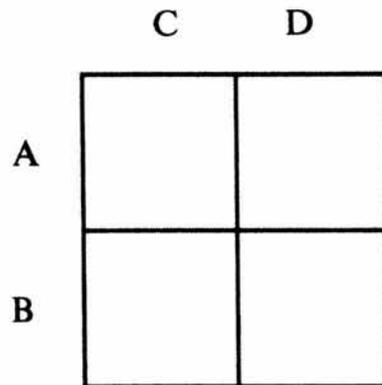
The second aim is reinterpreted. If axiomatic set theory is not a theory of manifolds, then what is it a theory of? The key notion here, that of an individual which is a *representative* of a manifold, is also suggested by Stenius, but again the idea goes back further. This aim will not be pursued in any great detail, though outlines of a theory embodying such a reinterpretation are sketched in § 6.

The third aim arises out of the other two. Because of the power of most systems of axiomatic set theory, sufficient power in most cases to serve as a foundation for finite and transfinite arithmetic and almost all of the rest of mathematics, sets have been massively over-used by logicians and philosophers in ontological investigations, and made to do service for such diverse entities as numbers, properties, relations, orderings, functions, propositions, facts, theories, worlds, persons, material bodies, higher-order objects, and so on. If, as I believe, a theory of manifolds serves to outline the ontology of nothing but manifolds (whatever they are manifolds of), then much of the set-based ontology of modern philosophy represents theft rather than honest toil, and the work for the most part remains to be done. The third aim, which is accomplished if the first two are, is not to *do* this work but to clear the decks for it. The substantive work left to be done is formal ontology, of which manifold theory comprises a small but not insignificant part.

§ 1 Classes as Many and as One: Historical Remarks

Introductory textbooks on set theory usually contain on page 1 a sentence like this: 'A set is a collection of things regarded as a single object', with a warning not to take 'collection' to imply any kind of physical bringing-together of the things in question. Such a conception raises in extreme form the ancient problem of the one and the many. Something which is a collection, i. e. many, is also one. It is completely specified by its members but is distinct from them, even when it has only one. The intelligibility of this kind of stipulation has in recent years been questioned, above all by Black and Stenius. This essay is in large part a development of the line of thought opened up in particular by Black, who first formulated with clarity the view that sets are to plural terms as individuals are to singular terms. The one-many problem cannot be avoided by taking a set to be the whole comprised of its members, their mereological sum. In the first place, such a sum does not in every case exist, or at least

it is not clear that it must.² In the second, even when such a whole does exist, it will not usually satisfy the fundamental principle of sets, the principle of extensionality: that sets are the same if, *and only if*, they have the same members. For two different collections may comprise the same whole when summed: this divided square



is the sum of the top half and the bottom half as well as the sum of the left half and the right half. We must accordingly distinguish the sum $A + B$ from the set $\{A, B\}$, for $A + B = C + D$ but $\{A, B\} \neq \{C, D\}$.³ Sums are wholes, and thus also individuals, whereas manifolds with more than one member are not individuals but pluralities.⁴ The whole-part relation $<$ is a relation between individuals, and must therefore be distinguished from the membership relation \in .^{5, 6} A mereological approach to classes has always held attractions for those of an anti-Platonist turn of mind. Goodman indeed *defined* Platonism, somewhat idiosyncratically, as the acceptance of sets. I would suggest that nominalist scruples about sets as abstract entities, 'high-brow' sets, might be to some extent assuaged by the use of manifolds, which are 'low-brow' sets, no more abstract than their members.

In the face of the successful advances of axiomatic set theory since Zermelo's first axiomatization of 1908, logicians have for the most part simply put aside or ignored the problem of one and many.⁷ If we look back, however, to the origins of set theory, when intuitions were perhaps fresher and less apt to be moulded by a tradition, we find a much greater awareness of the issue. In particular I wish to show how the problem made itself felt to three great set theorists: Cantor, Russell and, more recently, Bernays.

§ 1.1 Cantor

Cantor explicitly regarded a set (*Menge*) as a comprehension into a whole (*Zusammenfassung zu einem Ganzen*) of a plurality (*Vielheit*) of different objects. After the appearance of Burali-Forti's paradox, and in view of his own proof that since a set has more subsets than members, the impossibility of there being a universal set (sometimes called Cantor's paradox), he came to realize that it cannot be the case that to all pluralities (*Vielheiten*) there should belong a set (*Menge*). Those pluralities which can be comprehended into wholes he called *consistent*, those which cannot he called *inconsistent*.⁸ That Cantor can accept contradictions with such remarkable equanimity is due not to his being a working mathematician with better things to do⁹ but to his having on hand the distinction between sets and pluralities. He even went so far as to outline principles for deciding when pluralities can be comprehended and when they cannot, foreshadowing later developments.¹⁰ It is unfortunately not clear what the nature of this 'comprehension' is, but the important point for our purposes is that Cantor apprehended a distinction between sets and other individuals on the one hand and pluralities on the other, turning it to good use when the paradoxes were discovered.

§ 1.2 Russell

A distinction analogous to that between *Mengen* and *Vielheiten* is to be found, independently of Cantor, in Russell's early work of genius, *The Principles of Mathematics*. The importance of this work for our purposes lies in the circumstance that Russell, in the first flush of his enthusiasm for realism, was more sensitive to fine distinctions than he was to be later, after the success of the theory of descriptions in depopulating much of his universe spurred him to further reductions. In § 70 of this work, Russell distinguishes between a *class as one* and a *class as many*. He regards this as an 'ultimate distinction'.¹¹ What is especially interesting and important is that Russell, like Cantor, does not introduce the distinction for the express purpose of providing a way out of the antinomies, although, like Cantor, he does thereafter avail himself of the distinction for this purpose.¹² The immediate need for the distinction arises rather in connection with the argument put forward by Peano and Frege for distinguishing singleton sets from their members. The argument

goes thus: suppose we invariably identify x with $\{x\}$. In the case where x is itself a class with more than one member, since $\{x\}$ has just one member, and $x = \{x\}$, it follows that x both has just one member and more than one, a contradiction. Russell takes this argument to establish rather that we should not be tempted to identify classes as one with classes as many: 'the many are only many and are not also one.'¹³ For $\{x\}$ can only have one member which is itself a class if ' x ' denotes a class as one, while x can only have many members if ' x ' denotes a class as many. The Peano-Frege argument turns on an ambiguity and so founders: there can be, from Russell's point of view, no case where a class as many is a *member* of another class, since only *individuals* (Russell's *terms*) can be members. The difference between individuals (including classes as one) and classes (as many) is one of *type*.¹⁴

The distinction blocks Russell's paradox in that the non-self-membered classes comprise only a class as many: there is no corresponding class as one.¹⁵ This is essentially the same as Cantor's approach.

Russell even anticipates, though somewhat unclearly, Black's view on the crucial role of plural reference:

In such a proposition as 'A and B are two' there is no logical subject: the assertion is not about A, nor about B, nor about the whole composed of both, but strictly and only about A and B. Thus it would seem that assertions are not necessarily *about* single subjects, but may be about many subjects.¹⁶

Russell adverts to the use of 'and' to form what he calls 'numerical conjunctions' or 'addition' of individuals: '*A and B* is what is denoted by the concept of a class of which *A* and *B* are the only members.'¹⁷ Russell sways between denying that plural propositions can have genuine logical subjects and allowing that they do.¹⁸ He is also vague to the point of unintelligibility about the status of classes as many:

In a class as many, the component terms, though they have some kind of unity have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from being many.¹⁹

Russell admits that he cannot find any individual like Frege's *Wertverlauf* (a word Russell felicitously translates as 'range') which is distinct from his own class as one. But whereas Frege's range is designed to obey the principle of extensionality, Russell's classes as one are mereological sums, and so do not.²⁰

Nevertheless, without a single object to *represent an extension*, Mathematics crumbles . . . But it is exceedingly difficult to discover any such object, and the contradiction proves conclusively that, even if there be such an object sometimes, there are propositional functions for which the extension is not one term.²¹

Russell's exasperation is clear. He is for the most part happy to regard the extension of a concept under which more than one thing falls as a class as many, but feels, in part under Frege's influence, the need for individuals to do the work of extensions. Why should mathematics crumble without these? Russell offers one brief example, and another reason is not hard to find. Firstly, consider a simple combinatorial problem: How many ways can m things be selected from n things, without regard to order, where $m < n$? The answer, $n!/m!(n-m)!$, is usually taken as the cardinality of the set of subsets of cardinality m of a set of cardinality n . This requires that we treat sets as members of other sets, i. e. use classes as one. But, on Russell's mereological view of classes as one, should any of the m things be a part of one of the others, the wrong answer would result. So we appear here to need something like Frege's range, which obeys the principle of extensionality while still being an individual. Secondly, Russell, like Frege, wants to give the logicist account of numbers as classes of equinumerous classes, but again if only classes as one can be members of other classes, the only number which could be thus defined is the number one, and that still remains a class as many. Russell again badly needs Frege's ranges: a number can then be taken as the range of the concept *equinumerous with M*, for suitable choice of concept (or range) M (I am ignoring Frege's difficulties about referring to concepts). But Russell's paradox has blocked for ever the unconditional guarantee of such handy individuals. Rather than admit the bankruptcy of logicism, Russell prefers to look to the complications of type theory, which he outlines in the second Appendix to *Principles*. From here on, the distinction between classes as one and as many ceases to play a role, and the whole idea of a class is eventually dropped in favour of a reduction to propositional functions.²²

§ 1.3 Bernays

Between the wars Zermelo's initial axiomatisation of set theory was modified and improved by various writers. Skolem made more precise

Zermelo's vague notion of a definite property, and Fraenkel proposed the Axiom Scheme of Replacement in place of Zermelo's Axiom Scheme of Separation, to allow unrestrictedly for transfinite ordinals. With Miriamoff's suggestion that all sets should be founded, so that for no set s_0 would there be an infinite descending sequence $\dots s_{k+1} \in s_k \in \dots \in s_2 \in s_1 \in s_0$, the shape of what is now always called ZF set theory was complete.²³ In 1925 von Neumann reinjected Cantorian ideas into set theory with a distinction between *sets* and *classes*.²⁴ This allowed axiom schemata to be replaced by axioms, and set theory was for the first time finitely axiomatized.

In a series of papers from 1937 to 1954, Paul Bernays developed von Neumann's treatment along somewhat similar lines. Bernays' treatment is usually taken as a mere variant of the approach of von Neumann, and the similar approach of Gödel 1940: the three are run together under the title NBG set theory. But there is a difference between the treatment of Bernays and those of von Neumann and Gödel which is quite crucial from our point of view. Whereas von Neumann and Gödel both regarded sets as classes, namely those classes which can be elements of other classes, even though Gödel, for example, used different faces for set and class variables, Bernays keeps sets and classes distinct from one another, allowing smaller and more tractable classes to *correspond* to sets. In his development of this theory he uses its finite axiomatization property to interpret it in a two-sorted first-order predicate calculus, with sets and classes comprising the different sorts, and two different primitive membership relations.²⁵ This is usually regarded as an unnecessary nuisance, since it complicates the symbolism and the treatment of mathematics, and the expedient of identifying sets with their corresponding classes is usually employed. But the thinking behind Bernays' treatment is clearly motivated by philosophical rather than mathematical considerations, as the following passage shows:

The two kinds of individual [sc. sets and classes], as well known, can in principle be reduced to only one kind, so that we come back to a one-sorted system . . . However it might be asked if we have here really to go as far in the formal analogy with the usual axiomatics. Let us regard the question with respect to the connection between set theory and extensional logic. As well known, it was the idea of Frege to identify sets with extensions (Wertverläufe) of predicates and to treat these extensions on the same level as individuals. That this idea cannot be maintained was shown by Russell's paradox.

Now one way to escape the difficulty is to distinguish different kinds of individuals and thus to abandon Frege's second assumption; that is the method of type

theory. But another way is to give up Frege's first assumption, that is to distinguish classes as extensions from sets as individuals.²⁶

Bernays' axiomatic theory of sets and classes consists in showing how to attain full freedom of set construction according to the intuitive principles laid down by Cantor, with sufficient power to derive classical mathematics, while avoiding the paradoxes. It thus constitutes a fulfilment of the idea, sketched, but never followed through, by Russell in Appendix A of *Principles*, of allowing unrestrictedly classes as extensions of propositional functions, while employing certain individuals as *Ersatz* extensions, Frege's ranges, in order to develop classical mathematics.²⁷

This is not to suggest that Bernays regarded classes as manifolds in our sense, that is, as 'many's' of individuals. Rather, he regarded them as individuals, though apparently as less substantial individuals than sets: useful fictions, perhaps.²⁸ However, he does speak of sets as *representing* classes. It would not therefore do excessive violence to at least the letter of his views if we were to regard classes, the extensions of predicates, as manifolds in our sense, and sets as individuals which are taken for mathematical purposes to represent the more tractable classes. Such an idea will be pursued further in § 6 below.

§ 2 Linguistic Phenomenology of Plural Reference

Plural reference was already introduced in the previous essay. Plural terms are expressions apt for referring to more than one thing at once. They contrast not with general, but with singular terms. A singular term is an expression apt for referring to, denoting or designating an individual. As the name suggests, it is (in Indo-European languages at least) usually inflected or otherwise modified for number, and when the subject of a clause, the main verb of the clause will usually agree with it in number. General terms, such as 'man', 'hooded crow', 'horse with a wooden leg' etc. are unfortunately so called, in that both general and singular terms might be assumed to be subsumed in a single category of terms. But I believe Frege was right in considering such general words and phrases (which I shall henceforth call *common noun phrases* (CNPs), where Frege called them 'concept words') as being inherently predicative rather than referential, although I do not consider CNPs to

be *simply* predicates, but rather to occupy a position intermediate in various respects between predicates and terms, constituting in fact a basic category of expression distinct from terms.²⁹

Singular terms should be contrasted rather with plural terms, which are also referential rather than predicative. Whenever we use a term, the syntax of English and many other languages compels us to treat the term as either singular or plural, and modify it accordingly. This can on occasion be a nuisance in ordinary discourse, and would be a considerable drawback in formulating an artificial language for logical purposes. The problem of how to deal formally with modification for grammatical number will be considered in the next section.

As outlined in the previous essay, plural terms fall into the same sub-categories as singular terms, namely proper names, descriptions, demonstrative phrases and pronouns, as well as having sub-categories not available, for obvious reasons, to singular terms, namely term lists. We have already seen how Bolzano, Russell and others drew attention to the possibility of forming term lists by using the word 'and' any number of times, flanked by that number plus one terms. The usual method of writing out a name for a finite set, as ' $\{a,b,c\}$ ' etc., constitutes, for those not under the impression that this expression denotes a new abstract unit, another feasible way of forming plural terms. Plural terms, like singular terms, may be different in sense and yet still designate the same things, while plural demonstratives, pronouns etc. are indexical in exactly the same way as their singular counterparts. Just as a singular term ('that man', 'the owner of 34 High Street'), may be used to refer to different individuals on different occasions of its use, so a plural term ('those men', 'John's children', etc.) may on different occasions of its use refer to different manifolds of things.

A plural term like 'the people in this room' is to be sharply distinguished from the (plural) CNP 'people in this room'. Whether singular or plural, CNPs are not terms. This difference is both syntactic and semantic. Semantically, CNPs do not of themselves make definite reference to things. Apparent exceptions, like 'People in this room have been smoking', can be set aside. In this case, although the CNP occurs alone as subject of the sentence, it is not a referential use, but quantificatory. The sentence means something like '*Some* people in this room have been smoking'. It is doubtful whether there is an exact logic for the quantificatory uses of CNPs in subject position. Sometimes, as in the above case, the meaning is existential, at others, as in 'Men are mortal', it

is universal, at others, as in 'People went home at midnight' it is probably majoritive, meaning something like 'Most people . . .', and in yet other cases ('Tigers have four legs', 'Gentlemen prefer blondes') the meaning is one of vague typicality, perhaps requiring some new kind of typicality-operator. Syntactically the difference varies according to language. In English, terms, unlike CNPs, may not be preceded by articles, demonstrative pronouns or quantifier phrases. In other languages the conventions differ: e. g. in Italian proper names require the definite article. In some languages, such as the Slavonic ones where articles are lacking, the difference is certainly less marked, and it might be preferable to regard the term/CNP distinction as somewhat parochial, especially in view of the long tradition of grouping proper and common nouns together in the one category of *name*.³⁰ Nevertheless, while the syntactic distinction may vary in strength according to language, the semantic distinction, between a nominal expression which is, and one which is not, marked for *definiteness*, whether this marking is morphological, syntactic, contextual or whatever, is one which cannot be ignored. As it happens, we shall not employ anything like common nouns in the formal treatment of § 4, but this is essentially a move away from ordinary language to the predicate/variable language of orthodox logic, where there is no CNP category.³¹

Mention must be made of collective nouns, like 'class', 'group', 'set', 'collection', 'aggregate', 'herd', 'flock', 'bunch' and the like. If *c* is a collective noun and *d* is some other CNP then '*c* of *d*s' is a CNP in the singular, yet we rightly regard such phrases as 'this flock of sheep' as referring to many individuals, though not one at a time. In the terminology of the previous essay, the expression may designate each of many sheep without *subdesignating* any of them, i. e. without containing a subterm designating any one. But, unlike a plural expression like 'these sheep' the expression 'this flock of sheep' is syntactically singular, and the question naturally arises whether we have here a singular term or an *ostensibly singular plural term*.³² Much of the appeal of the trinitarian concept of sets, whatever there is to be said against it, derives from the familiarity of cases where we use a grammatically singular expression to somehow characterise a plurality of individuals. The very words 'set', 'class' etc. are themselves collective nouns used for just this purpose. Do collective noun phrases refer to new, higher-order individuals, constituted by but distinct from their members, or do they simply refer to manifolds of individuals? I believe that, if we consider carefully, we

shall see that they do neither, although they share in part the behaviour of singular terms and in part the behaviour of plural terms referring to a manifold. To facilitate the discussion, I shall annex the word 'group' to describe what such terms refer to, or rather to describe, somewhat weakening my claim, what many or most of them seem to refer to. This answer is important, since on acceptance of it rests my suggestion that set theory (manifold theory) is a poor tool for ontological research (since most groups are not manifolds).³³

Two facts about groups have to be noticed: we shall then be clearer as to what a group is. Firstly, when we use a collective noun, we never, or hardly ever, use it without an accompanying CNP, linked to it (in English), by 'of'. We have classes of degree, sets of cutlery, clumps of trees, herds of cattle, collections of stamps and so on. In other words, groups are always groups *of* individuals, often of a specified sort. Secondly, to take up a point noticed by Stenius,³⁴ what makes certain individuals belong to a group is almost always more than their being several of the kind comprising the group. Not just any plurality of trees constitutes a clump, and not just any plurality of postage stamps constitutes a philatelic collection, and so on. The members of the group are linked, tied, connected or associated in some way. To borrow the terminology of Husserl from the first essay, between the members of the group there subsist various *foundation relations*. Such relations may take many forms. It may be that all the individuals in the group have a common relation to one thing, as for example when all the grapes in a bunch are connected, directly or indirectly, to one stem, or all the bees in a swarm are following the one Queen. It may be, alternatively, that the ties are simply relations holding between or among the members of the group, as for instance all the trees in a clump are relatively close to one another and further from other trees, or all the stars in a galaxy are relatively strongly attracted to one another gravitationally, as well as being closer to one another than to stars in other galaxies.

These facts distinguish groups in general from mere manifolds. For it is characteristic of a manifold that its members may be anything whatever. They need have no intrinsic ties or foundation relations: the only tie they need have is the purely extrinsic one of all being designated by one and the same term. Since we may form terms arbitrarily by listing, it is not surprising that the most bizarre bedfellows may be together in a manifold. Most of the manifolds we take any interest in are, mercifully, not of this kind. But the most important feature distinguishing most

groups from manifolds is this: the identity of a manifold is purely parasitic upon the several identities of its members: it obeys the principle of extensionality: manifolds are the same iff they have the same members. Groups on the other hand obey neither the 'if' nor the 'only if' part of this condition. A group may have different members at different times, and still be the same group. If a single tree is felled in a clump, the clump is diminished, but not destroyed. Likewise, if a new tree grows up in the clump, it is the same clump, but now augmented. Similar remarks may be made about other groups. Just as individuals, at least, those individuals which we call substances, may gain or lose *parts* to some extent without loss of identity, so groups may gain or lose *members* without loss of identity. I still attend concerts by the same orchestra I heard ten years ago, although the personnel has changed appreciably over that time. It is in this respect that groups are analogous to individuals, at least to individual substances, meriting the term 'higher-order objects' for groups. On the other hand, groups differ from individuals in being multiply constituted: a group may not *be* a manifold, but at any one time its members constitute a manifold. It is for this reason that the members of a group may be referred to using a plural term: we may refer to the trees in a clump as 'these trees', for example. It may be that the line between groups and individual substances is not a sharp one: a herd of cattle is certainly a group, and a multicelled organism like a man is certainly an individual, but certain colonies of insects resemble single organisms in various ways such as specialisation of role and balance of functions, while there is genuine dispute as to whether sponges are colonies of single-celled organisms or multicelled organisms of a different kind from most.³⁵

Because a group is not constituted solely by its members, but is the group it is in part because of the foundation relations among them, one and the same manifold of individuals may constitute, either successively or simultaneously, more than one group. To revert to the example of orchestras: in the days of the Empire, three of the orchestras of Vienna had the same personnel: when they played in the Court Chapel they were the Orchestra of the Court Chapel, when they played in the pit at the opera they were the Court Opera Orchestra, and when they played symphony concerts in the Musikverein they were the Vienna Philharmonic. Similarly two committees may have exactly the same members, yet not be one committee. In cases where two groups have the same members, we shall say they *coincide*. Because different groups have different persis-

tence conditions, two groups may first coincide and then not, or vice versa.

It would be as wrong to regard groups as mere successions of manifolds as it would be to regard individual substances as mere successions of 'genuine' individuals.³⁶ Just as we may regard individuals which can neither gain nor lose parts without ceasing to exist as a limiting case of individual substances, which can gain or lose parts, so we may regard manifolds as limiting cases of groups: those whose identity is exhausted by that of their members. In such circumstances the 'foundation relation' is the purely formal one of being just these several individuals and no others, although when we *refer* to a manifold using a plural term, this adds the weak extrinsic tie mentioned above.

Given that manifolds are groups obeying the principle of extensionality, manifold theory is powerless to describe the constitution of groups not obeying this principle, just as mereology is powerless to explain the nature of an individual which may gain or lose parts.³⁷ Nevertheless, it will not be wasted effort to develop the formal theory of manifolds, any more than it is a wasted effort to develop a mereology. Groups are, or are usually, 'many-fold', and a formal theory of pluralities will serve to show something of the logic of plural reference, as well as linking up more obviously with traditional set theory, where extensionality is always obeyed. To this end further aspects of the use of plural terms, those especially relevant to the basic notions of such a formal theory, should be mentioned.

Firstly, there is identity. We have spoken rather glibly of the identity of groups, but we need to be assured that there can be genuine identity predications involving plural terms. Sentences like the following:

The men in this room are John and Henry

resemble singular identity sentences in two important respects. Firstly, like singular identities, and unlike copulative sentences, the terms flanking the verb may be commuted without loss of sense, indeed without loss of truth (or falsity). Secondly, the logical properties of identity: reflexivity, symmetry, transitivity and intersubstitutibility in all extensional contexts *salva veritate* apply in the plural case also. There are apparent counterexamples to this last claim. Suppose John and Henry are the men in this room. Then while we may say

- (1) The men in this room are few
- (2) Max is not one of the men in this room,

the following sentences are less acceptable:

- (3) ?John and Henry are few.
- (4) ?Max is not one of John and Henry.

These facts do not however amount to a refutation of the proposition that intersubstitutability applies to plural terms. Sentence (1) is somewhat idiomatic as it stands: it would be far more acceptable to say the same thing by

- (5) There are few men in this room.

In this case, there is no plural to be substituted, and the problem vanishes. On the other hand, if by 'few' we mean something fairly definite, say, 'less than ten in number', then even if we accept (1) at its face value as containing a plural term, and tantamount to something like

- (6) The men in this room number less than ten (men),

then substitution gives

- (7) John and Henry number less than ten (men).

The readiness to drop the second occurrence of 'men' in (6) but not (7) may be explained by its having already occurred once in (6). (7) seems to me no less acceptable than (6). In case (2), again, if (2) is tantamount to something like

- (8) Max is not a man in this room

then the problem vanishes, whereas if we accept (2) at face value as containing a plural term, as I am more inclined to than with (1), then we may look on (4) as merely pragmatically or conversationally deviant, in that it is not usual to use different names for one person in close proximity, so that the need to make assertions like (4) does not often arise. Nevertheless, cases when assertions like (4) would be both apt and true are not hard to imagine: for instance

- (9) Tully is one of Vergil and Cicero, but not one of Plautus and Livy.

If sentences like 'The men in this room are John and Henry' are not plural identities, it is hard to see what they could be.

I thus take it as established that identity has a sense which is shared

between singular and plural identity propositions, involving the syntactic and logical properties mentioned above. Plural identities need not entail singular identities either: for instance

(10) John's parents are the two oldest inhabitants of the village

entails nothing about which parent is oldest and which is second oldest.³⁸

Next, there is membership and inclusion. We must distinguish between sentences like

(11) John is a man in this room

(12) These cows are brown

where the predicate does not involve a plural term, from those such as

(13) John is one of the men in this room

(14) These cows are among the cows owned by Brown

where the predicate does contain a plural term. The copulas 'is', 'are' (and their equivalents in other tenses) cannot be considered candidates for the vernacular equivalents of the ' \in ' and ' \subseteq ' of set theory, which are binary predicates, flanked by terms. For ' \in ' the nearest equivalent in English is 'is one of' or 'is (one) among', e. g. 'John is among the winners of Olympic Medals of 1964'. The nearest equivalent of ' \subseteq ' similarly appears to be 'are among' or 'are some of'.³⁹

Now the difference between 'is one of' and 'are some of', or between 'is among' and 'are among', appears to be no greater in principle than that between 'is' and 'are' or 'runs' and 'run': one of grammatical number. While this is only a linguistic point, and does not bear directly on set theory, it is worth recalling that the Peano-Frege distinction between membership and singular inclusion was not always regarded as commonplace. Some of the most notable logicians of the last century such as Schröder and Dedekind did not make the distinction, while in Leśniewski's Ontology the distinction between singular inclusion ' $a \in b$ ' and strong inclusion ' $a \sqsubset b$ ' is merely that the former is false if 'a' is not a singular name,⁴⁰ otherwise the two are equivalent. It is worth recalling also that the Peano-Frege argument rests on the assumption that sets can be *members* of other sets even when they contain more than one member, a view which Russell was, at first, not ready to accept at face value, and in which we agree with him.⁴¹ The case for there being a distinction

of *type* between individuals and pluralities thereof rested for Russell on there being certain predicates which applied to individuals which did not apply to pluralities, and vice versa.⁴² But 'apply to' is ambiguous. It can mean that the predicate may be predicated *truly* of the subject, or that it can be predicated *significantly* of the subject. Only the second yields evidence for a distinction of type. The first suggests, trivially, only that there are some predicates true of individuals and not true of pluralities, and, if incorporated into a logical system yields a type-free system like that of Leśniewski.

There are indeed predicates which are, at least, never true of individuals. Most obviously, there are the plural number-predicates, like 'are seven in number'.⁴³ Less obviously, there are predicates such as 'meet', 'disperse', 'surround', and those derived from relational predicates, like 'are shaking hands', 'are similar', 'are cousins'. (I have put these in the plural: it is of course trivially true that a predicate in the singular cannot *correctly* follow a plural subject and vice versa, but the underlying verbs 'be shaking hands' when used in the singular sometimes have the dereferentialised sense 'be shaking someone's hand', although this can hardly be said of all the predicates mentioned here.) Some of the predicates, like 'disperse' and 'surround', may be used in a grammatical singular number, but in such cases they apply not to individuals but to masses of stuff, as 'The fog is dispersing', 'Water surrounds the house'.

The existence of such predicates might be used to justify the introduction of type distinctions. But if one prefers to say that sentences like

?John surrounded the fort

?The cow dispersed to various parts of the field

are not nonsense but simply and necessarily *false*, as I confess I am inclined to do, although I have not usually succeeded in getting agreement on this, then the same examples may be used to stake a stronger claim for the legitimacy of plural reference. Whether plural reference is always eliminable in favour of singular, or singular reference together with quantification, is not in any case the main point. I certainly believe that even if plural reference is in principle eliminable, it would be at least highly inconvenient to actually eliminate it, and maybe practically impossible. However, I am not claiming its ineliminability or its practical indispensability, merely its existence and usefulness. It is not as if eliminating plural reference brings ontological economy. Manifolds do not exist over and above, or even alongside, individuals. A manifold is simply

one or many individuals. A manifold exists if and only if at least one individual exists.⁴⁴

Looking at the question of membership and inclusion from the point of view of plural reference, the semantic condition:

True iff everything designated by the subject term is also designated by the object term (where the subject term is the one before, and the object term the one after, the relational predicate)

applies equally to membership, inclusion, and indeed identity, which can be considered a limiting case of inclusion. Were we to replace 'everything' by 'anything' in the above condition, this would also let in the case where the subject term is empty.⁴⁵

§ 3 Problems of Formalisation

The phenomenon of agreement or concord in syntax arises whenever expressions in certain syntactic categories fall into subcategories in such a way that even when two expressions are of compatible categories, that is, categories such that when expressions from them are combined, the result is syntactically connected,⁴⁶ (as for instance adjective and common noun, term and verb), there are still restrictive rules, usually called *selection restrictions*, governing which combinations are to count as well-formed. When such rules are violated, we get the most obvious examples of bad grammar, as *'This books', *'They smokes' etc. Many languages utilise selection restrictions in connection with grammatical number, distinguishing singular from plural, among terms, verbs, nouns, adjectives etc., and sometimes a three-way distinction between singular, dual and plural (more than two).

It is interesting that in typed languages like that in *Principia Mathematica* the restrictions on forming formulas may be regarded as selection restrictions, with each category of expression: term, predicate etc. being divided into denumerably many different syntactic subcategories. The extreme inconvenience of such restrictions may be seen by the frequent resort made in describing typed languages to the device of typical ambiguity.

Despite our introduction of plural terms, it would be similarly inconvenient for us to have a formal language employing selection restrictions with respect to grammatical number. Suppose we had a language

with singular terms s, s', \dots and plural terms t, t', \dots and predicates P, P', \dots (one-place) and R, R', \dots (two place), and we require further that predicates always occur modified for number, so that if P, R , etc. are the unmodified predicates, then \dot{P}, \dot{R} , etc. are in the singular and \bar{P}, \bar{R} , etc. are in the plural. Suppose further that selection restrictions operate as follows: the number of a predicate is to be the same as that of its first argument. This procedure would resemble closely the practice in many natural languages. Now consider how we should state that a binary relation R is symmetric: in more orthodox formal languages it would go

$$(\forall xy)(xRy \supset yRx)$$

whereas in the suggested language it would go

$$(\forall s s' t t') ((s\dot{R}s' \supset s'R\dot{s}) \& (s\dot{R}t \equiv t\dot{R}s) \& (t\dot{R}t' \supset t'R\dot{t}))$$

where we have shortened it somewhat by using a biconditional in the middle conjunct. Similar encumbrances would accompany all other generalisations: consider how formidable the formula stating the transitivity of R would look, for instance. This problem has not arisen hitherto because formal languages have invariably employed only singular terms.

One possible weakening would be to make the modification of predicates optional rather than compulsory, allowing concord to be used for highlighting certain predications. This would however necessitate the postulation of equivalences like $(s\dot{R}t \equiv sRt)$, and would not result in the total disappearance of selection restrictions anyway, since $s\dot{R}t$ would still be ill-formed. Once modification of predicates becomes optional, it seems arbitrary to stop there; better to drop modification altogether. Predicates would then be *neutral* as between singular and plural, and there are no longer any selection restrictions. This does not stop us from continuing to divide the category of terms into singular and plural. We can reflect the difference between the singular 'is' of identity and the plural 'are' of identity by having two different identity predicates, '=' for singular identity and ' \approx ' for plural identity. Then rather than extract a *syntactic* penalty when a singular term flanks the plural predicate or vice versa, and declare the result ill-formed, we shall extract a *semantic* penalty, and declare the result *false*. This is preferable to declaring it meaningless, for we should then have to decide how to deal with well-formed formulae lacking truth-values in compounds, and this is a messy

affair⁴⁷ which we can quite easily avoid. However, in line with the view put forward in the previous section that there is a sense to the identity concept which is independent of the distinction between singular and plural, we shall employ an identity predicate ' \simeq ' which is *neutral* as between singular and plural, in that it may be flanked by either singular or plural terms and still be true.⁴⁸

Suppose that s and s' are singular terms designating the same individual. What is the status of the ostensibly plural term ' s and s' '? It has the form of a list, yet, if it designates anything, it designates just one thing. Two policies are open here I think, only one of which will be pursued in the next section. We could take a list like this to be a plural term, as its syntactic form suggests, but because it is not *semantically* plural, regard it as empty, having no referent. On the other hand, we could regard it as having as referent the same individual as it subdesignates with both its subterms. This then poses the question whether a redundant list of this kind should be counted singular or plural: syntactically it looks plural, whereas semantically it looks singular. However, as in the case of predicates, there is no reason to regard the singular/plural distinction as exhaustive of all the kinds of terms there might be. It is highly expedient to employ *neutral terms*, which are neither singular nor plural syntactically, but can be either singular or plural semantically. In practice such neutral terms are far more useful than strictly plural ones, and in the formal language developed in § 4, neutral terms will be employed extensively. I do not know of any neutral terms in natural language, though there is no reason in principle why natural languages should not employ them. In the language of § 4, all term lists will be neutral: plural term lists could be used, but are not.

We must next decide policy on empty terms. There is a vast literature on the problem, since Russell first proposed the theory of descriptions. It would be impossible to review in detail which course is the best to adopt, although I believe that the course I shall adopt is both the best and the most natural. Russell's procedure for singular definite descriptions is to allow them as terms only where existence and uniqueness have been proved. Against this it has been objected that this complicates the notion of term in that there is then no general decision procedure as to which expressions are terms, and involves a further complication in case the uniqueness and existence formulas are not categorically, but only conditionally derivable.⁴⁹ Frege on the other hand proposed arbitrarily assigning a referent to every term which would otherwise be emp-

ty.⁵⁰ This procedure seems, indeed is, artificial, and a better and more natural alternative is available. While recognising, with Frege, the syntactic affinity between names and descriptions, that is, regarding them all as terms, we regard empty terms as being simply terms which do not designate anything. It has sometimes been said that an expression may denote without denoting anything. This seems absurd to me. Provided only that we are able to handle sentences containing empty terms in our logic, then there is no need to assign artificial referents to empty terms. It is the great merit of free logic that it allows empty terms to be handled for logical purposes without requiring the development of three-valued truth-tables for the connectives. Of the free logics available, the one which appears to me to have the best philosophical justification is one which allows some formulas containing empty terms to be true, others to be false, and yet others to have no fixed truth-value on an interpretation. This is the sort of free logic developed by Lambert and van Fraassen.⁵¹

The possibility will be admitted of all terms, whether singular, neutral or plural, being empty. This indeed could hardly be otherwise if the logic is to be free of existence assumptions in remaining valid for the empty domain, when all terms are perforce empty. There will in fact be a standard empty term ' \wedge ', and this will be neutral. Furthermore, the neutral identity predicate ' \simeq ' will be allowed to hold truly when flanked by empty terms. Indeed, to keep the system extensional, we shall require that if ' a ' and ' b ' are both empty, that ' $a \simeq b$ ' is true. Consideration of the intended semantics of ' \simeq ' shows why this is so: ' $a \simeq b$ ' is to count as true iff whatever is designated by ' a ' is designated by ' b ', and vice versa, and this is vacuously so when both terms are empty. The standard empty term ' \wedge ' therefore plays a role like that of ' \emptyset ' in standard set theory, except that \emptyset is usually taken to exist: ' $(\exists x)(x = \emptyset)$ ' is a theorem in standard set theory, while we shall have as a theorem ' $\sim E \wedge$ ', where ' E ' is the existence predicate. This appears to me to be a considerable intuitive advantage of the present theory of manifolds: set theorists were once wont to deny that there was an empty set, or apologize for it as a 'convenient fiction',⁵² though in latter days they have become more brazen about asserting its existence. Systems of pure set theory, without *Urelemente*, admit indeed nothing but \emptyset and the various sets compounded therefrom according to the axioms, which, from the point of view of intuitive considerations, is a total retreat from reality. We can, with a sensible logic allowing the manipulation of empty terms, gloss the notion of a "convenient fiction" not by reluctantly admitting the entity as having a shad-

owy sort of existence, but by allowing that a term may be highly useful and yet still be empty. To sum up: there is no empty manifold. But, skirting paradox, we might say, extensionality ensures that there are not two distinct empty manifolds.

When empty terms are admitted, but bivalence is retained, there are various ways in which quantifiers and variables may be employed. The first is to take variables as ranging over both actual and possible objects, with universal and particular quantifiers meaning roughly 'for all (actual and) possible', 'for some (actual or) possible', and an existence predicate separating the actual from the merely possible.⁵³ This line appears not only blatantly extravagant ontologically, but also rests on the dubious notion of a purely possible individual. A second possibility would be to follow Leśniewski and allow what has been called 'unrestricted quantification'⁵⁴, so that for instance both ' $\forall xPx \supset Pa$ ' and ' $Pa \supset \exists xPx$ ' would be true even where 'a' is an empty term. I shall not follow this line here, since I believe it gives a non-standard meaning to the quantifiers.⁵⁵ However, I shall return in § 5 to a comparison with Leśniewski's ontology, which could be readily interpreted as a calculus of neutral terms. The approach followed here maintains allegiance to the maxim that to be is to be the value of a bound variable, allowing all parameters to be empty, but not variables. This is the approach of free logic, as exemplified by Lambert and Van Fraassen.⁵⁶

Since we shall hold to Quine's maxim on existence, and manifolds with more than one member may exist, it is only consistent to allow variables to range not only over individuals, but also over manifolds of individuals. We shall accordingly employ three kinds of variable, corresponding to the three kinds of term: singular, plural and neutral. We shall allow these all to be bound by quantifiers and the description operator, so the intended meaning of sentences and terms where variables other than singular ones are bound must be spelt out in the next section. It would be possible to dispense altogether with singular and plural terms and variables without loss of expressive power, but we have not done so in the present exposition, since the motivation for the introduction of neutral terms etc. was that there could be plural ones. Having established that plural terms do indeed play a role in natural languages, it would be somewhat ungrateful to banish them completely from our formal language, although the price to be paid for keeping in the three sub-categories of term and variable is, as we shall now see, a certain complication of the formalism.

§ 4 Axiomatisation of Manifold Theory

In this section we shall be concerned to present axioms for a theory of manifolds, remarking as we go on the intended interpretation of the axioms. No formal semantics will be set out, nor will any metamathematical results concerning the system be proved. Such tasks lie in the future. The first task is to make the basic ideas more familiar.

We shall speak about an object language without being too concerned as to what it actually looks like: all axioms and rules will be characterised metalinguistically, using schematic meta-axioms. Definitions will be regarded as semantically motivated metalinguistic abbreviations.⁵⁷ So, if a and b are terms such that ' $a := b$ ' is a definition, a and b are automatically intersubstitutable, and ' $a \simeq b$ ' is a metatheorem. Similarly, if A and B are formulas such that ' $A := B$ ' is a definition, A and B are automatically equivalent, and ' $A \equiv B$ ' is a metatheorem.

Primitive Symbols of the Metalanguage

The following constant symbols are used:

Connectives: 1-place: \sim ; 2-place: \supset .

Quantifier: \forall .

Determiner: 1.

Predicates: 2-place: \simeq ; \in .

Punctuators: ($;$).

The following metavariables range over expressions of the kind listed:

Terms: Singular: s, t, s', t', s'', \dots etc.

Plural: m, n, m', \dots

Neutral: q, r, q', \dots

All terms: a, b, c, a', \dots

Variables: Singular: w, x, w', \dots

Plural: h, k, h', \dots

Neutral: u, v, u', \dots

All variables: y, z, y', \dots

Predicates: 1-place: P, P', \dots

2-place: R, R', \dots

(predicates of greater adicity will not be considered.)

Well-Formed Formulas: A, B, C, A', \dots

Formation Rules

Those expressions which are terms and well-formed formulae (wffs) are specified by a double recursion.

A *term* is either a singular term, or plural term, or neutral term.

Singular Terms comprise singular parameters (if there are any in the object language), singular variables, and singular descriptions, and nothing else.

Plural Terms comprise plural parameters, if any, plural variables, plural descriptions, and nothing else.

Neutral Terms comprise neutral parameters, if any, neutral variables and neutral descriptions, and nothing else.

Descriptions have the following forms: singular: ιxA
plural: ιhA
neutral: ιuA

where A is any wff. Descriptions in general therefore have the form ιzA .

Terms may therefore be divided into singular, plural and neutral or into parameters, variables and descriptions.

Wffs comprise atomic and compound wffs, and nothing else.

Atomic wffs have the forms: Pa ; aRb , where a and b are any terms.

Compound wffs have the forms: $\sim(A)$; $(A \supset B)$; $\forall zA$, where A and B are any wffs, atomic or compound.

The usual definition of free and bound occurrences of variables within terms and formulae will be understood. An *open formula* is one containing at least one free variable occurrence. A *closed formula* is a formula in which all occurrences of variables are bound. Assuming that the variables are given some linear alphabetic ordering, then if A is any wff, the *universal alphabetic closure* of A is that wff obtained from A by binding all the free variables remaining within it with universal quantifiers, working outwards in alphabetic order. If A is closed, then it is its own closure. In the following, the expression ' $\vdash A$ ' will mean 'the universal alphabetic closure of A is a theorem'.⁵⁸

' $A(b/a)$ ' will designate that formula obtained from A by substituting occurrences of b for all occurrences of a , while ' $A(b//a)$ ' will range over all formulae obtainable from A by substituting occurrences of b for occurrences of a in all, some or none of the places where a occurs. In each of these definitions it is assumed that if A contains a well-formed

part of the form $\forall aB$ in which the term b occurs free, this part is rewritten with a variable not otherwise occurring in A . We shall also dispense with parentheses wherever possible, following the conventions of Church.⁵⁹ Thus

$$A \supset B \supset C := ((A \supset B) \supset C)$$

$$A \supset . B \supset C := (A \supset (B \supset C))$$

and we shall continue this practice when other connectives are introduced.

The constants ' $\&$ ', ' \vee ', ' \equiv ' and ' \exists ' are defined in the usual way in terms of ' \supset ', ' \sim ' and ' \forall '.

Meta-axioms for Predicate Logic

- a1 If A is a tautology of propositional calculus, $\vdash A$.
- a2 $\vdash \forall z(A \supset B) \supset . \forall zA \supset \forall zB$
- a3 $\vdash A \supset \forall zA$, where z is any variable not free in A .
- a4 $\vdash \forall zA \supset A(y/z)$, where z is free in A , and y is of the same subcategory as z .
- a5 If A is a theorem and $A \supset B$ is a theorem then B is a theorem.
(Modus ponens)

These axioms are of a form which is familiar in free logics. They differ from axioms for predicate calculus with existence assumptions by not having such theorems as ' $\forall xPx \supset Ps$ ' or ' $Ps \supset \exists xPx$ ', since the *dictum de omni* axiom a4 is restricted to the case where a variable is replaced by another variable.

The difference between these axioms and those for normal free logic lies of course in the fact that we have three kinds of variables. It should be made clear how these work. If D is any non-empty domain of interpretation, then an assignment of values to variables in D assigns individuals to singular variables, manifolds with at least two members to plural variables, and manifolds with at least one member (i. e. manifolds in general) to neutral variables. Of course, if D is a singleton, no values can be assigned to plural variables, and only individuals to neutral variables. In similar fashion, if there are any parameters in the object language, an interpretation over D assigns individuals or nothing to singular parameters, pluralities (by which I mean manifolds with at least two members)

or nothing to plural parameters, and manifolds or nothing to neutral parameters. The difference between parameters and variables thus consists in the possibility of parameters being empty even on non-empty domains.

So ' $\forall xPx$ ' means that the predicate P applies to all individuals, ' $\forall hPh$ ' means that P applies to all pluralities, and ' $\forall uPu$ ' means that P applies to all manifolds. From this it will be seen that care should be taken not to mix variables of different categories carelessly. This is catered for by the restrictions in a4. It also seems evident that both $\forall uA(u/a) \supset \forall xA(x/a)$ and $\forall uA(u/a) \supset \forall hA(h/a)$ should be metatheorems, since whatever is true of all manifolds should also be true of all individuals and of all pluralities. Such a metatheorem, $\forall uA(u/a) \supset \forall zA(z/a)$ is indeed forthcoming, but in order to prove it further axioms are needed which will serve to link the roles of the various subcategories of variables.

Meta-axioms for Identity

$$a6 \vdash a \simeq a$$

$$a7 \vdash a \simeq b \supset . A \supset A(b//a)$$

The predicate ' \simeq ' is the neutral identity predicate, holding between terms a and b just when they designate the same manifold. The familiar properties of symmetry and transitivity are readily derivable. On the other hand, the extensional property, that $a \simeq b$ when both a and b are empty, is not derivable from a1–a7, and has to be ensured by further axioms. It must be noticed that $a \simeq a$ holds for *all* terms: in this lies its usefulness. However, with identity and quantification on hand, we could readily define an existence predicate and various other identity predicates.

In free logic, existence is usually defined in terms of identity and quantification, and we could proceed thus:

$$Ea := \exists u(u \simeq a)$$

with singular and plural existence defined as follows:

$$E!a := \exists x(x \simeq a)$$

$$E!!a := \exists h(h \simeq a)$$

Notice that here the distinction between the three subcategories of variable allows us to define three closely related predicates. It will turn out that these are not the only ways in which existence could be defined, but they are intuitively appealing to some, in that they represent the maxim: to be is to be identical with something.⁶⁰ It should be here noted that there are indeed systems of free logic in which the existence predicate is present but the identity predicate is lacking. In such systems, not only is existence not defined in the usual way; it can be shown to be undefinable (I owe notice of this to Karel Lambert).

A neutral identity predicate which does not hold between empty terms may be defined thus:

$$a \cong b := Ea \ \& \ (a \simeq b)$$

and singular and plural identities as follows:

$$\begin{aligned} a = b &:= E!a \ \& \ (a \simeq b) \\ a \approx b &:= E!!a \ \& \ (a \simeq b) \end{aligned}$$

while yet further predicates would cater for the cases where we allow singular-or-empty, and plural-or-empty terms. The predicate ' \simeq ' is however taken as basic here because of the familiar properties represented by a6–a7, preserving the analogy with singular identity in our chosen system of free logic.

Inclusion

As indicated in § 3, membership is to be regarded as singular inclusion. To reflect this, we choose as primitive the predicate ' \in ' of non-empty neutral inclusion. Its intended interpretation is as follows: $a \in b$ is true just in case (i) a is non-empty and (ii) every individual designated by a is also designated by b .⁶¹ This is captured by the first axiom a8 below. In addition we now introduce the principle of extensionality in a9: manifolds are the same if they have the same members (the converse follows from a7 and the quantification axioms). For technical reasons, I prefer to define the existence predicate E not in terms of identity, as in the previous subsection, but in terms of inclusion as follows:

$$Ea := \exists x(x \in a)$$

and $E!$ and $E!!$ similarly (but these predicates will not be used.) This could be expressed as: to be is to comprise at least one individual. It applies to individuals as well as to pluralities. That it amounts to the same thing as the previously suggested definition can be seen only if we grant that variables range over things that exist, singular variables ranging in ones, plural variables in twos or more, neutral variables in ones or more. This is the import of a10, which comes in three instalments for the three subcategories of variable. To formulate the condition for singular variables, we need to be able to say when exactly one individual satisfies a given condition. In fact we shall give a more comprehensive definition, which enables us also to say what it means for exactly one individual, exactly one plurality, or exactly one manifold, to satisfy a condition.

First, we define 'at least one' trivially as follows:

$$\exists_1 zA := \exists zA$$

and now we define 'at most one':

$$\exists^1 zA := \forall z \forall y (A \& A(y/z) \supset y \simeq z)$$

where it is a condition that y and z belong to the same subcategory. This sort of definition, without the complication about subcategories, is in any case already familiar from ordinary first-order predicate logic with identity. We now simply define 'exactly one' as usual as 'at least and at most one'.

$$\exists_1^1 zA := \exists_1 zA \& \exists^1 zA$$

The sense of $\exists^1 xA$ will be familiar already, but what of $\exists_1^1 hA$? This says: there exists exactly one plurality such that A , whereas $\exists^1 uA$ means: there exists exactly one manifold (whether singular or plural) such that A . Suppose for tax purposes an apartment block is divided into *households*, some of which are individuals, others *families*. Then $\exists_1^1 xA$, $\exists_1^1 hA$ and $\exists^1 uA$ respectively correspond to saying something like: there is exactly one individual/family/household in the block such that . . .

Meta-axioms for Inclusion

$$\text{a8} \quad \vdash a \in b \equiv Ea \ \& \ \forall x(x \in a \supset x \in b)$$

$$\text{a9} \quad \vdash \forall x(x \in a \equiv x \in b) \supset a \simeq b$$

$$\text{a10a} \quad \vdash \exists_1^! x(x \in w)$$

$$\text{b} \quad \vdash \exists_2 x(x \in h)$$

$$\text{c} \quad \vdash \exists x(x \in u)$$

To understand a10b the numerical quantifier \exists_2 must be defined. This is done in the obvious way:

$$\exists_2 z A := \exists z \exists y (A \ \& \ A(y/z) \ \& \ \sim (y \simeq z))$$

where y and z must be of the same subcategory.

We can define different inclusion predicates in terms of the notions introduced up to now. Of these, the most interesting are singular inclusion, or membership, and inclusion which holds even when the subject terms is empty:

$$a \in b := E!a \ \& \ a \in b$$

$$a \subset b := \sim Ea \vee a \in b$$

Some ready metatheorems following from these axioms and definitions tell us e. g. that existence and self-inclusion come to the same thing: $Ea \equiv a \in a$, that everything is emptily or genuinely self-included: $a \subset a$, and that, when singular terms are in question as subjects, inclusion and membership amount to the same thing: $s \in a \equiv s \in a$. A metatheorem which will be of interest in the next section is the following

$$\vdash a \in b \equiv \exists u(u \in a) \ \& \ \forall u(u \in a \supset u \in b) \\ \ \& \ \forall uv (u \in a \ \& \ v \in a \supset u \in v)$$

where it will be noted that instead of using a singular variable and neutral inclusion predicate to express existence, we may equivalently use a neutral variable and the singular inclusion predicate. This suggests that we could manage with slimmer resources: neutral terms alone. That this is so is shown by these metatheorems, which express the 'ubiquity' of neutral terms:

$$\vdash \exists u(u \approx z)$$

$$\vdash \exists a \equiv \exists u(u \approx a).$$

Descriptions

When descriptions are introduced into a system with plural and neutral terms we must consider how their sense is to be specified. With singular descriptions we already know how to gloss 'the x such that', which (when completed, e. g. by ' x is in this room') is the nearest equivalent in a language without common nouns to a natural language description like 'the man in this room'. If we look at plural descriptions in natural language, such as 'the men in this room', then it is clear that something is comprised in the manifold of such men as a member if and only if it is a man in this room. However some predicates, unlike 'man in this room', apply to pluralities. Consider 'meet' for instance. A sentence of the form ' a met' can only be true if ' a ' is a plural term. Corresponding to this verb we get as plural description something like 'those who met'. But clearly an individual can truly be said to be among those who met: the manifold designated by 'those who met' is a plural manifold, but like all manifolds is comprised of individuals, even if the predicate used does not itself apply to the individuals individually, so to speak. Consider a complicated plural description like 'those who met either in the dining room or in the lounge'. Clearly an individual belongs to the manifold so designated iff he is one of those who met in either (or both) of those places: we could specify which manifold is designated here by giving a list of *individual names*. Suppose, for example, that John, Fred and Jim met in the dining room, while Mike, Sam and Fred met (later) in the lounge. Then those who met in either the dining room or the lounge are John, Fred, Jim, Mike and Sam. (This is a good example of a plural identity sentence at work.)

Neutral descriptions may be understood then as covering both the individuals such that . . . and the pluralities such that The list of English monarchs comprises not only those who ruled alone, but also William and Mary, the joint monarchs.

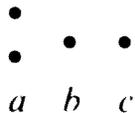
A pocket-size example will show the different kinds of description at work. Consider various collections of dots drawn on the page, and envisage them as being in a procession proceeding from left to right across the page. Let the one-place predicate ' R ' be interpreted as 'forms a rank

in the procession.' Any one or more dots in line abreast form a rank. Names will be assigned to the ranks: a token of each name appears below the rank in question. Where a rank consists of only one dot, the name is singular, and where it consists of more than one dot the name is plural. Then in the first procession



$\iota xRx \simeq b$, while $\iota hRh \simeq a$, and $\iota uRu \simeq a$ and b . We shall in fact write lists with terms between braces: a precise definition will follow. Here $\iota uRu \simeq \{a,b\}$. The term ιxRx is not empty, because there is a unique singleton rank, namely b . The term ιhRh is likewise not empty, because there is at least one rank with more than one member. The term ιuRu embraces all those things which are in any rank.

In the second procession



the term ιxRx is empty, because there is no unique singleton rank, while $\iota hRh \simeq a$ and $\iota uRu \simeq \{a,b,c\}$. It must not be thought here that because the term ' $\{a,b,c\}$ ' contains three atomic subterms that the manifold thereby designated contains three individuals: here it contains four. It is also useful to have an expression marking out the manifold consisting of *all* individuals falling under R . We shall use the familiar notation $\{x\}A$, for this purpose also, to stress further the analogy with normal set theory. The definition is this:

$$\{x\}A(x/a) := \iota u(\exists! x(x \in u) \& A(u/a))$$

and in the second procession $\{x\}Rx \simeq \{b,c\}$.

In the third procession



both ιxRx and ιhRh are empty, since there are no plural ranks and there is no unique singleton rank. Here $\iota uRu \simeq \{x \mid Rx\} \simeq \{a, b\}$.

In the fourth procession

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ a & b \end{array}$$

both ιxRx and $\{x \mid Rx\}$ are empty, while $\iota hRh \simeq \iota uRu \simeq \{a, b\}$. If there could be such a thing as a null procession, all the descriptions would then be empty.

Descriptions, especially neutral descriptions, add greatly to the expressive power of the language. They enable us to define a great many constants in a way which makes the resulting theory begin to resemble more familiar set theory.

Firstly we define the universal manifold:

$$\forall := \iota u(u \simeq u).$$

and then, by analogy, we can define

$$\wedge := \iota u \sim (u \simeq u)$$

It will transpire that for every empty term a , $a \simeq \wedge$ is true. This is the extensionality principle mentioned earlier. Another metatheorem will be $Ea \equiv a \in \forall$: to be is to be comprised among the things there are. Only in the trivial interpretation over the empty domain is $\wedge \simeq \forall$. Like the night in which all cows are black, in the empty domain all terms, including the most comprehensive one \forall , are empty. The terms \wedge and \forall play a role similar to that of 0 and 1 in a Boolean algebra. The difference is that in Boolean algebras the zero element exists. However there are interesting analogies with Boolean algebras, which will be touched on briefly in the next section.

It is now possible to go ahead and define the usual Boolean operators of union, intersection and complement.

$$\begin{aligned} a \cup b &:= \iota u(u \in a \vee u \in b) \\ a \cap b &:= \iota u(u \in a \& u \in b) \\ a - b &:= \iota u(u \in a \& \sim \exists v(v \in u \& v \in b)) \\ -a &:= \forall - a \end{aligned}$$

These definitions could alternatively have been given using the notation $\{x\}A$ just introduced. In this guise they look more familiar, especially if we use the singular inclusion predicate. As it is, the following are forthcoming as metatheorems:

$$\begin{aligned} \vdash a \cup b &\simeq \{x\}x \in a \vee x \in b\} \\ \vdash a \cap b &\simeq \{x\}x \in a \& x \in b\} \\ \vdash a - b &\simeq \{x\}x \in a \& x \notin b\} \end{aligned}$$

From these definitions we may form arbitrary finite unions and intersections, and, because of extensionality, the usual Boolean identities and equivalences hold. The use of lists in normal discourse corresponds to expressions like $\{a, \dots, c\}$ in ordinary set theory, and we shall have an equivalent. We define term lists inductively as follows: if a is any term, then a is a term list, and if d is any term list, then a, d is a term list. We get terms from term lists by surrounding by braces, and the resulting terms are defined as follows:

$$\begin{aligned} \{a\} &:= a \\ \{a, d\} &:= \{a\} \cup \{d\} \end{aligned}$$

This may seem like cheating, but it isn't. Given the motivation of the previous section, lists designate the individuals designated by each term in the list, whether it be singular or plural. We shall follow the convention that all term lists are neutral terms. Finite lists turn out to be indistinguishable from finite unions. This is in contrast to orthodoxy in set theory, but is motivated by the phenomenology of plural reference. It means that there are in the present theory no manifolds of manifolds distinct from manifolds of individuals. This point was defended as intuitively justified in the previous essay. In practice what it means is that manifolds do not stack up in an infinite hierarchy of types or ranks, but remain single-storied. This ought to appeal to the lovers of desert landscapes. So any expression formed out of terms by nesting lists to any finite depth may be replaced by a one-dimensional list, erasing all the braces except the outermost. Other cherished distinctions from orthodox set theory are casualties also. Firstly there is the distinction between a and $\{a\}$, as the last definition shows. In general it is only true that $a \in \{a\}$ when $E!a$, a is a singleton. Where a is a singleton, not all set theorists

distinguish the element from the singleton set. As we mentioned before, Dedekind did not, and Cantor was not firm either way, while, in recent times, Quine has regarded the distinction as dispensable.⁶² The view that $a \in \{a\}$ only when a is a singleton embodies what I believe is right about Russell's distinction between classes as one and classes as many: that the only classes as one that exist are singleton classes!

One of the most powerful devices for generating sets in Zermelo's theory was the power set axiom. However, if we look at what must be our equivalent, the power set of a is $\{u(u \in a)\}$. This turns out to be nothing but a , again, as our informal motivation would suggest. It is a metatheorem that $\vdash a \simeq \{u(u \in a)\}$. For example if a is the pair $\{s, t\}$, then the power set of a is $\{\{s\}, \{t\}, \{s, t\}\}$. (There is no null manifold: even if we defined the power set in terms of ' \subset ' rather than ' \in ' the result would be the same, in any case.) Now, recalling that where braces are nested, we may remove all but the outermost, this manifold is revealed as $\{s, t, s, t\}$, which is simply an unnecessarily long way of designating $\{s, t\}$.

It might be thought that we are now crippled in terms of expressive power. How, for instance, can Russell's combinatorial problems be stated, and what is the status of the assertion that if a manifold has g members, then it has $2^g - 1$ submanifolds (minus 1 because there is no null manifold)? Firstly, we can talk about all the manifolds that satisfy a certain condition, rather than all the manifolds belonging to a (higher-order) manifold. There is nothing to stop us from making assertions about all pairs, for instance. But if we try to assemble the manifolds together into a single manifold in order to be able to 'handle' them (surely a manifestation of prejudice in favour of the singular), we shall find that we lose the original manifolds, getting landed simply with their union manifold. The manifold of all pairs is (assuming at least two things exist) simply \vee . The use of conditions instead of higher-order manifolds does bring a loss of expressive power however if one is not prepared to quantify over conditions. It may be said, however, that any ontological commitments incurred in quantifying over predicates is not lost when one trades predicates for sets: it simply reappears in a different form.⁶³ In any case, the paradoxes show that not every condition can earmark a distinctive individual as the corresponding set: this was where Bernays entered the fray in 1937.

We shall not develop number predicates or numerical quantifiers in detail, although it is clear from the definitions of \exists_1 , \exists_2 etc. and the discussion of the previous paper how finite number predicates and numeri-

cal quantifiers can be defined. But it is important to realize that we may define similar looking but different numerical quantifiers by using different subcategories of variable. For instance, $\exists_2 xRx$ and $\exists_2 hRh$ do not mean the same thing, and neither means the same as $\exists_2 uRu$. Consider the processions examples again. $\exists_2 xRx$ means that two individuals are ranks: this is true in the second and third cases, false in the first and fourth. $\exists_2 hRh$ means that exactly two pluralities are ranks: this is only true in the fourth case. $\exists_2^u Ru$ means that exactly two manifolds are ranks. This is only false in the second case. So we are quite able to say that

$$\exists_2^u x(x \in a) \equiv \exists_{2_{\exists_2^u}} u(u \in a)$$

and indeed, given recursive definitions of the numerical predicates, the result could be proved as a metatheorem by mathematical induction.

Combinatorial problems such as would have warmed the cockles of Russell's heart could drop out of the system as metatheorems. For instance, a football manager with a squad of thirteen players has to pick an eleven to take the field. He can select any one of 78 different possible teams: but it would be surely only a matter of patience to prove the following as a metatheorem of the calculus of manifolds:

$$\exists_{78}^u u(u \in a \ \& \ \exists_{11} x(x \in u) \ \& \ \exists_{13} x(x \in a))$$

(I have not the patience.)

Having different styles of variable and accordingly different senses for numerical quantifiers also enables us to put a firmer gloss on the contention, made in the previous essay, that number predicates, when applied to individuals, have senses analogous to the sense they have when applied to pluralities or manifolds in general. The analogy comes out in the common form of the definitions of numerical quantifiers despite the use of different subcategories of variable. In this way the informal motivations of the previous essay link up with the formal treatment of this one. This distinction enables us easily to do the work which Stenius suggests requires a procedure he calls "second-order counting".⁶⁴ Indeed, it is more flexible, since it allows us to count arbitrary finite numbers of manifolds, not just those which are submanifolds of a given manifold.⁶⁵

But suppose, to revert to our example, that the football manager, not content with knowing how many teams he can pick, wishes to know how many ways he can slot his selected players into the eleven available positions, and arrives at the (I hope) correct answer of 39,916,800. How can this be expressed in terms of the 11 players, maybe the 2047 submanifolds thereof, without sets of sets? Surely it is here that we need sets of sets, or, as Stenius uses, arbitrary representative individuals to go proxy for sets. I am not convinced. Certainly simply considering the 11 players and submanifolds of them will never advance us to the relatively astronomical figure of 11!; but I do not think we are in this case counting men, or groups of men, at all. We are computing possible *ways* of slotting eleven men into eleven positions. This is the same as the number of different ways we may pair any two disjoint collections of eleven, or, speaking mathematically, the number of different bijections between disjoint sets of eleven. I would suggest that expressing combinatorial problems in terms of sets of sets, or sets of sets of sets, is merely a convenient device, and does not represent the ontology of combinatorial problems at all.

Meta-axioms for Descriptions

- a11a $\vdash E_{1x}A \equiv \exists_{1x}A$
- b $\vdash E_{1h}A \equiv \exists hA$
- c $\vdash E_{1u}A \equiv \exists uA$
- a12 $\vdash E_{1z}A \supset \forall y(A(y/z) \supset y \in {}_{1z}A)$, where z is either neutral or of the same subcategory as y .
- a13 $\vdash s \in {}_{1z}A \supset \exists u(s \in u \ \& \ A(u/z))$
- a14 $\vdash \sim E_{1z}A \supset {}_{1z}A \simeq {}_{1u}(u \neq u)$
- a15 $\vdash a \simeq {}_{1z}(z \simeq a)$, where z is either neutral or of the same subcategory as a .

The three instalments of a11 present the conditions on the existence of manifolds designated by descriptions. a12 and a13 tell us about the membership of manifolds designated by such descriptions when they exist, while a14 “identifies” all empty descriptions in the way suggested by § 3. This treatment is most suited for mathematical applications, though the possibility of varying the axioms for other applications, e. g. in considering the logic of fiction, is not to be ruled out without further

consideration. The final axiom a15 states an identity not otherwise derivable. This in fact makes a6 derivable as a metatheorem.⁶⁶

The axiom a11c is quite a powerful one, and simulates union axioms in orthodox set theory after this fashion: If for instance Pu states a condition in one free variable on manifolds, then so long as at least one manifold satisfies the condition, the union of all such manifolds exists. Conversely, if such a union exists then at least one manifold satisfies the condition. That $\cup Pu$ is in effect a union can be seen by considering its membership conditions, using a12–a13. By a12, if $\cup Pu$ exists, then any manifold satisfying the condition is included in it, and by a13, any individual which is a member of $\cup Pu$ is a member of some manifold satisfying the condition. Notice that the individual need not itself satisfy the condition: this should be clear from the examples given before. In general, we cannot infer either that the union $\cup Pu$ itself satisfies the condition: if P is the predicate ‘is a pair’ then in any world containing three or more individuals, the manifold of pairs is not a pair.

This sort of consideration may put one in mind of Russell’s paradox. It is worth seeing how it fails to arise in the present theory. All singleton manifolds are self-membered, and all pluralities are not. The manifold of non-self-membered manifolds is $\cup(u \notin u)$. In a domain with less than two members, this does not exist. In one with two or more members, it exists, and is identical with \forall . Now in such domains certainly \forall exists and $\forall \notin \forall$, but this does not entitle us to infer that $\forall \in \forall$: merely, and harmlessly, that $\forall \notin \forall$. $\forall \in \forall$ only when the domain has only one member, and then $\cup(u \notin u) \approx \Lambda$. The paradox simply does not arise, for precisely the reason originally suggested by Russell: there is a gulf between one and many.

In general $P(\cup Pz)$ only in the case where $\cup Pz$ is a singular description which is not vacuous, although cases arise with other subcategories: for instance, in domains with at least two members, the manifold of manifolds with more than one member is itself a manifold with more than one member, viz. \forall . We can indeed prove as a metatheorem the following general principle of comprehension:

$$\vdash s \in \{x | A(x/a)\} \equiv A(s/a)$$

though again the manifold $\{x | A(x/a)\}$ only satisfies the condition $A(\xi)$ when exactly one individual satisfies it, and this individual is the manifold.

Axiom of Choice

While the foregoing meta-axioms delineate a system with deceptive power, the following principle appears to be independent of them, and yet intuitively satisfactory, especially in the form given.

$$\text{a16 } \vdash \exists u A(u/a) \& \forall u \forall v (A(u/a) \& A(v/a) \& u \neq v \supset u \cap v \simeq \Lambda) \\ \supset \exists u \forall v (A(v/a) \supset \exists ! x (x \in (u \cap v)))$$

What the axiom amounts to is this: if $A(\xi)$ is any condition in one variable satisfied by at least one manifold, and such that any two distinct manifolds satisfying it are pairwise disjoint, then there exists a manifold intersecting each manifold satisfying the condition in a single element. This is sometimes known as the weak or disjoint choice principle. It is hard to see how it could be questioned. In this form, the axiom is not really about choice or selection in any real sense: it is about the existence of certain manifolds.

In his 1908 paper on set theory,⁶⁷ Zermelo used the principle in just this disjoint form, although the pairwise disjoint sets were not those satisfying a condition, but those belonging to a set of sets. Notice that in our case we do not need to state that the sets be non-empty: this is taken care of by the variables, which range only over manifolds that have members. Zermelo's original 1904 proof of well-ordering⁶⁸ uses not this disjoint principle but the principle which he called in 1908 the General Choice Principle: that any set of non-empty sets possesses a choice function. In his 1908 paper on well-ordering he again uses the General Principle as premiss, but, as if by way of placation, assures that the General Principle is but a consequence of the Disjoint Principle,⁶⁹ while in his paper on the foundations of set theory the General Principle is derived as a consequence of his axioms. Now Zermelo gives the appearance of regarding the Disjoint Principle as more likely to secure acceptance from the sceptical, while proving that it is just as strong as the General Principle. But what he in fact shows is that the General Principle follows from the Disjoint Principle together with the other axioms of Zermelo's set theory. These include assumptions about set existence, especially the power set and infinity axioms, which are much stronger than we have employed. By these means, Zermelo is enabled to trade in arbitrary sets of sets for equinumerous but pairwise disjoint sets of sets, using pairs consisting of one element and one set. Such means are not

here available, nor indeed can the strong General Principle be formulated as one of set existence, which makes it appear rather different in kind from the Disjoint Principle. It has indeed been suggested that Zermelo's axiomatisation was motivated less by a desire to avoid paradoxes as to gain acceptance of the well-ordering theorem, in which the axiom of choice serves of course as premiss.⁷⁰

The interesting question left unanswered by this is whether, in the presence of weaker though still intuitively justifiable assumptions as to set existence, the General Choice Principle does not turn out to be stronger than Disjoint Choice.⁷¹

§ 5 Some Comparisons

The following remarks assess in broad outlines the affinities of the system presented in the previous section. In many respects the ideas, despite some obvious departures from current practice, represent a return to an older tradition, not fully distinguished in its time from general logic, namely that tradition running from Leibniz through Boole, Peirce and Schröder to Husserl, Löwenheim and Leśniewski, a tradition to be distinguished sharply from that running from Frege, Peano and Russell through to modern predicate logic on the one hand and from Cantor and Frege through Zermelo to modern axiomatic set theory on the other. Despite Russell's initial clarity about classes, he soon forsook that path in favour of a reduction of classes to propositional functions.

In many ways the present system is similar to Schröder's application of his calculus of identity and subsumption to domains taken in extension. Schröder developed a type theory of sorts.⁷² Church has suggested that this was essentially a substitute for a difference between set membership and set inclusion.⁷³ But Schröder introduces the type-like distinction rather to avoid paradoxes. These arise, in my opinion, through a lack of adequate understanding of the difference between a predicate's being applicable to a thing and a thing's being included in a domain,⁷⁴ together with an inability to handle empty terms. Schröder uses the symbol ' \in ' for subsumption: there are in his earlier, type-free system in addition the two domains 0 and 1 such that $0 \in a \in 1$ for every domain a . But Schröder does not distinguish between every element of one domain being an element of another, i. e. subsumption, and a subject's be-

ing characterised by a predicate. Thus he regards ' $0 \in a$ ' simply as signifying that '0 is subject to every predicate a '. Hence, if a predicate determines a domain a , then since $0 \in a$, that predicate applies to 0. So, in considering the predicate 'is equal to 1', Schröder regards the class of classes (domain of domains) satisfying this predicate as comprising just 1 and 0. But since 0 is subject to this predicate it follows that $0 = 1$, and all distinctions collapse: the night in which all cows are black.⁷⁵ Hence, Schröder concludes that classes of classes should be distinguished according to level from classes of individuals. That is indeed one way out, although it did not appeal to Frege. Frege suggested that Schröder's *Gebietenkalkül* was really only a theory of part and whole, and that in such a case there could be no null entity 0. I agree with the view that *if* this is how we interpret Schröder's system, as a mereology, then an empty individual indeed is out of place. But our system, like Schröder's, is intended not as a theory of part and whole but as a theory of extensions of terms. In such an extensional approach to classes, there should likewise be no null class, as Russell saw.⁷⁶ But we can retain the usefulness of Schröder's 0 without regarding it as an entity, by the scrupulous use of empty terms. Schröder's paradox does not arise in our system, even though we do not distinguish classes of classes from classes of individuals, because while $\Lambda \subset a$ for every a it does not follow that Λ exists, nor, if a is $\cup Pu$ or $\{x|Px\}$, that $P \wedge$. *Pace* Frege,⁷⁷ the extension of a concept *does* consist of the things falling under it in the same way as a wood (as manifold, not group), consists of trees. Having on hand the concept of a manifold means that we can treat the extension of a concept as what it is: one or more individuals. An empty concept then is not a concept with an empty extension, if by this we mean that there is something, its extension, which happens to comprise no individuals. Rather, it is a concept without an extension.⁷⁸

The system of § 4 is a first-order system: we do not quantify over predicates. The differences, and complications, all arise from the introduction of plural terms and variables, with quantification over all manifolds, plural as well as singular. If the expressions involving terms other than singular are not employed, the remaining fragment is simply equivalent to a normal free logic, with ' \in ' equivalent to '=', where '=' differs from ' \simeq ' in not holding between terms which are empty.⁷⁹ On the other hand, the system cannot be proved consistent simply by interpreting all terms as singular (or empty), because it would be inconsistent when so interpreted: the axiom a10b would be interpreted as

$$\vdash \exists xw(x = h \& w = h \& x \neq w)$$

which is inconsistent.

The existing system of logic which our system most nearly resembles is Leśniewski's Ontology, sometimes called the calculus of names. In Leśniewski, names, like our terms, can designate one or more than one or they can fail to designate at all. On the other hand it is clear that Leśniewski's "names" comprise both what I should call terms, and common nouns. I had previously thought that the only possible interpretation of Ontology which made sense in terms of the sort of expressions to be found in natural languages was as a calculus of common nouns.⁸⁰ But it now seems to me that it can equally well be interpreted as a calculus of terms, whether these be singular, plural or empty. Leśniewski's calculus could be regarded as one involving solely neutral variables, with somewhat different principles and axioms governing quantification. In some Leśniewskian systems singular names are informally marked by use of capital letters, but this does not affect their substitutivity, which is why variables are all de facto neutral.

If we had adopted quantifiers without existential import, say \prod and Σ , such that $\Sigma uA := \sim \prod u \sim A$, subject to the axiom $\prod uA \supset A(r/u)$, where r is *any* term, empty or not, and axioms analogous to a2–a3, then we should have a ready way to interpret Leśniewskian expressions as follows:⁸¹

<i>Usual Leśniewskian Form</i>	<i>Interpretation</i>
$[a] \cdot A$	$\prod uA$
\in	\in
$\text{ex}(a)$	Er , or $\exists u(u \in r)$
$\text{sol}(a)$	$\exists^1 u(u \in r)$
$\text{ob}(a)$	$\exists \downarrow u(u \in r)$
\sqsubset (Strong inclusion)	\notin
\subset (Weak inclusion)	\subset
$=$ (Singular Identity)	$=$
\circ (Weak identity)	\approx

I have preferred to develop the calculus of manifolds in such a way that it is recognisably an extension of the usual predicate calculus involving only singular and empty terms. The introduction of identity as a primi-

tive by a6–a7 seems especially preferable, since identity has stronger claims to be a logical relation than inclusion or membership, in terms of which it is usually defined in Leśniewskian systems. However, despite its unusual treatment of quantifiers, Ontology can be said to embody a theory of manifolds, although these cannot be construed as sets in the usual sense.⁸² Ontology could claim to embody a skeletal theory of extensions of expressions, whether these be construed as common nouns or as terms, exhibiting the algebraic similarities between a calculus of nouns and a calculus of terms. In this it could be said also to belong in the Boole-Schröder tradition. I should be unwilling however to give up the view that there is a syntactic difference between terms and common nouns, despite their many semantic similarities. Such an identification erases many distinctions to be found in the syntax of natural languages, even though these distinctions may not be strictly necessary for logical purposes. An enlargement of the present theory could introduce common nouns and quantifiers and descriptors adjoining them.⁸³

I have several times mentioned the quasi-Boolean properties of the calculus of manifolds. It is instructive to see how we can interpret the axioms in certain Boolean algebras. This has the advantage of enabling us to trade in some of the more unusual features, such as empty and plural terms, with quantifiers binding variables other than singular, for an interpretation in which all terms are singular and quantifiers are as in a normal first-order theory, without even empty terms to worry about. The system is also thereby shown to be consistent relative to the algebras in which it can be interpreted. As the simplest of these are finite, this is a heartening claim. Let us consider the particular case first, and then comment briefly on more general interpretations.

Consider any subset of the positive integers consisting of all the divisors of a number which is square-free, in the sense that it has no divisors of the form p^2 , for $p > 1$. The smallest such set is $\{1\}$, but there is no largest, so we can have models of any finite cardinality 2^n , where $n \geq 0$. Let M denote any such set of divisors, with subscripts e. g. M_{30} , to denote particular cases. We may interpret predicate parameters as predicates defined over M , though we shall not in general be interested in arbitrary predicates. We interpret term parameters as follows:

Singular parameters are assigned either 1 or a prime number.

Plural parameters are assigned either 1 or a composite number.

Neutral parameters are assigned any number.

Variables are interpreted to range over M .

Quantifiers are interpreted as follows. If A is any condition in one free variable, suppose A' is the associated condition defined over M .

A formula $\forall xA$ is true on the interpretation iff A' is satisfied by all prime numbers in M .

A formula $\forall hA$ is true on the interpretation iff A' is satisfied by all composite numbers in M .

A formula $\forall uA$ is true on the interpretation iff A' is satisfied by all numbers greater than 1 in M .

It is to be noticed that all universal quantifications are vacuously interpreted as true on interpretation over the domain M_1 .

Descriptions are assigned values in M as follows:

If a single prime number in M satisfies A' , then ιxA is assigned that number, otherwise it is assigned the number 1.

If at least one composite number satisfies A' , then ιhA is assigned the lowest common multiple of all those composite numbers satisfying A' (which is in M , by choice of the sort of set M is), otherwise it is assigned 1.

If at least one number greater than 1 satisfies A' , then ιuA is assigned the l. c. m. of all those numbers that do, whether prime or composite, otherwise it is assigned 1.

Notice that, as with parameters, 1 is playing the role of a null manifold, prime numbers are playing the role of individuals, and composite numbers the role of pluralities.

We interpret primitive formulas involving ' \simeq ' and ' \in ' as follows:

A formula $a \simeq b$ is true on the interpretation iff a and b are assigned the same number by the interpretation (so we are interpreting ' \simeq ' as '=').

A formula $a \in b$ is true on the interpretation just in case a and b are assigned numbers a' and b' such that (i) $a' \neq 1$ (ii) a' divides b' .

From these it follows that a term $\{x|A\}$ is assigned the product of all the prime numbers satisfying A' (which, by construction, is in M), or else 1.

It may then be checked that on any such interpretation all the axioms a1–a16 come out as true, indeed logically true. a1–a6 are quite straightforward, being valid according to the usual principles of quantification and identity in any first-order theory. a8–a16 get interpreted as follows:

a8: if a' divides b' (where we shall assume that when we say one number 'divides' another, that it is also $\neq 1$), then $a' \neq 1$ and every prime factor of a' is a prime factor of b' .

a9: if a' and b' have the same prime factors, they are equal.

a10: prime numbers have exactly one prime factor, composite numbers have at least two, and numbers greater than 1 have at least one prime factor.

a11: these are conditions for the number assigned to a description to be $\neq 1$: that they are met can be seen by checking the conditions for assigning numbers to descriptions given above.

a12: if a number corresponding to a description is greater than 1, then every prime factor of any number meeting the associated condition A' divides this number. This is so by construction of the number as prime or l.c.m.

a13: if a prime number divides the number assigned to a description, then it divides some number satisfying the associated condition.

a14: if the number assigned to a description has no prime factors, then it is equal to 1.

a15: every number is equal to the product of all the numbers equal to itself, or, if prime, then equal to itself, or, if 1, then equal to 1.

a16: if some number greater than 1 satisfies a given condition A' , and all the numbers that satisfy A' are pairwise relatively prime, then there exists a number in M such that its common factor with every number satisfying A' is a prime number. That this is so is easily seen. Since the numbers satisfying A' are relatively prime in pairs, if we select one prime factor from each, say the smallest, then no prime is selected twice, and the product of all these primes is in M and satisfies the condition by construction.

The sets M_n form Boolean algebras under division as the partial ordering. This suggests that we could model the calculus of manifolds generally in any Boolean algebra. However, the algebra must satisfy certain conditions for the interpretation analogous to that given above for the finite algebras M_n to go through. This interpretation is a particularly straightforward and appealing one. Let B be any Boolean algebra, with distinguished elements 1 and 0, under the partial ordering \leq . Let us suppose further

(1) that B is *atomic*, i. e. for all elements $b \in B$, there is an element $a \leq b$ such that for all $c \in B$, $c \leq a$ implies $c = 0$ or $c = a$.

(2) that B is *complete*, i. e. for any non-empty subset A of B , a supremum $\sup A$ exists relative to \leq , that is, an element $s \in B$ such that (i) for all $a \in A$, $a \leq s$, and (ii) for all $s' \in B$ such that $a \leq s'$ for all $a \in A$, $s \leq s'$.

(3) that B is *distributive*, i. e. for every subset A of B which is not emp-

ty, and for every element $b \in B$, $b \cap \sup A = \sup\{b \cap a \mid \text{such that } a \in A\}$.

We can now sketch how interpretation in any such algebra B will go: since the details are similar to the finite cases M_m , we can be brief. Predicate parameters are assigned predicates defined over B , term parameters are assigned elements of B : singulars to atoms or 0, plurals to non-atoms or 0, neutrals to anything. Universal quantifications are true in these cases: singular variable bound: true iff every atom satisfies the associated condition, plural: true iff every non-atom $\neq 0$ satisfies it, neutral: true iff every element $\neq 0$ satisfies it. Descriptions are assigned elements of B as follows: if A'' is the set of elements of B satisfying the associated condition, then singulars are assigned 0 unless A'' is a singleton whose element is an atom, when this is assigned to the description. Plurals are assigned the supremum of the set of all non-atoms satisfying the condition, or else 0, and neutrals are assigned $\sup A''$ if $A'' \neq \emptyset$, or else 0. The completeness property assures that such a supremum exists where the set is not empty. The distributive property assures that suprema behave nicely in formulas. It corresponds to the following metatheorem of the calculus:

$$\vdash b \cap \sup A(u/a) \simeq \sup\{v \mid \exists u(A(v/a) \& v \simeq b \cap u)\}$$

The axioms for manifolds can then be verified to be valid for all such Boolean algebras. The Axiom of Choice is interesting, because while its proof was trivial in the finite case, to prove the validity of its interpretation in the general case, where B may be infinite, requires – unsurprisingly – the disjoint choice principle. For the interpretation comes to this: if $A \subseteq B$ is a subset not containing 0, such that for any distinct elements $a, b \in A$, $a \cap b = 0$, then there is an element $c \in B$ such that for all $a \in A$, $a \cap c$ is an atom. To see how it is proved, consider any such set A whose elements are pairwise relatively atomic. For each element $a \in A$, let $A(a)$ be the set of atoms $\leq a$. Since $0 \in A$, $A(a)$ is non-empty in each case, and, since if $a \neq b$ are both in A , $a \cap b = 0$, so $A(a) \cap A(b) = \emptyset$. Applying the disjoint choice principle to the $A(a)$, we select an atom from each. Let the resulting set of atoms be C . By completeness, $\sup C$ exists, and has the property that $a \cap \sup C$ is the selected atom in $A(a)$ for all $a \in A$, proving the result.

It is known that all Boolean algebras may be represented by an isomorphic algebra of subsets of some set, but in addition, if the Boolean

algebra is atomic, complete and distributive, in the senses given above, it is isomorphic to the algebra of all subsets of the set of atoms.⁸⁴ With this we come full circle.

I have also recently discovered that it is possible to interpret manifold calculus in ordinary whole-part theory. We simply interpret all terms as singular, and the relation ' \simeq ' as ordinary singular identity in a free logic, and the relation ' \in ' as the ordinary part-whole relation, so interpreted that only existents can be parts. The resulting calculus of individuals differs from that of Leonard-Goodman only in that it allows empty terms: a perfectly laudable difference, and that it is (according to axiom a10) *atomistic*, which is not necessarily so laudable. We can then interpret singular terms as designating *atoms*, plural terms as designating *non-atoms*, and neuter terms as designating all individuals, atomic or not. The only difficulty concerns the description operator, which does not readily generalise to the normal description operator. In fact, for plurals and neuters, the description operator represents the Leonard-Goodman *sum* or *fusion* operator. This difficulty can be removed by defining a new operator: let us confine ourselves solely to neutral terms here:

$$JuA := \imath u(A \& \forall v(A(v/u) \supset v \simeq u))$$

It is then the operator J which generalises under the mereological interpretation to the normal description operator. We can give axioms rather for J than \imath , which are symbolically exactly analogous to those for van Fraassen and Lambert's system FD₂, and then define \imath as follows:

$$\imath uA := Ju\forall x(x \in u \equiv \cdot \exists v(A(v/u) \& x \in v))$$

where we assume we have already defined ' \in ' through ' \in '. This now preserves perfectly the parallel with the fusion operator of the normal calculus of individuals. As to be hoped and expected, under the present interpretation, ' \wedge ' remains an empty term, unlike the case when we interpreted the calculus in Boolean algebras. This agrees naturally with the intuition that there are no null heaps, as Frege pointed out in his Schröder review, and the difference is perfectly congruous with Tarski's demonstration that mereology is Boolean algebra save for a Boolean zero. Of course this heartening symbolic parallel between the axiom systems in no way reduces manifolds to heaped individuals: far from it. In

an enriched language having both plural terms and a part-whole predicate, there would be things we should wish to say that we could not say if that were so, e. g. that no plurality is an individual, and that no mereological sum is a plurality. All the reasons I adduce in "Number and Manifolds" for rejecting the group theory of number here rise up again to refute the identification of manifolds with heaped individuals. In particular, the unheapability of such items as impossible possibilities, and the generally wider applicability of the notion of manifold than that of mereological sum, applications of which are predominantly confined to the physical sphere, speak loudly against such an identification. So the subsumption relation and the whole-part relation, whatever their algebraic similarities, must always be distinguished. A square built up out of four other squares has each of the four component squares as parts: it is their sum. But it is not identical with the *squares*, for there are four of them, and only one of it. Nor is a part of one of the squares (a proper part) one of the four squares, while it is part of the one square. So the relations 'is one of' and 'is part of' are quite different. Whoever appreciates this will have no problems about the one and the many. The main *axiomatic* difference between manifold theory and whole-part theory consists in the self-evidence of the fact that all manifolds consist of individuals, and the lack of self-evidence of the proposition that all individuals consist of atoms, i. e. Axiom a10. It is worth recalling in this connection the independence of the atomic hypothesis from general mereology in Leśniewski, while the requirement that manifolds always reach back to individuals recalls the necessity felt for Miriamoff's grounding axiom in ordinary set theory.

§ 6 Sets as Representatives of Classes⁸⁵

Stenius⁸⁶ suggests that the most plausible way to regard sets-as-things, as he calls them, or classes as one, is to regard them as individuals arbitrarily assigned to serve as representatives of, go proxy for, classes. He develops the idea that the relation of representation can be seen as a genuine relation between individuals in the domain, with the membership relation \in being considered as the converse of representation. In this way the formal results of the theory of sets may be preserved, without engendering the problems of the trinitarian conception of sets. The idea is

appealing: if all the mathematician wants is *some* object to do the job of sets, why not let him have an individual as proxy-object, subject simply to certain conventions on how to assign such proxies.

The idea is not new, however. Frege's *Wertverläufe* are precisely individuals which do service for functions, and have the added advantage of being saturated entities.⁸⁷ Frege's realism induced him to worry about what such *Wertverläufe* were: he was unable to take the conventionalist step of letting them be arbitrarily assigned subject to conventions. That some restrictions were necessary Russell found to Frege's cost. In a late paper of 1940, Löwenheim⁸⁸ suggested the 'Schröderisation' of mathematics by using individuals to represent classes, subject to restrictions analogous to those of axiomatic set theory to avoid paradoxes. Bernays reviewed the article quite favourably, which is not too surprising, since, as we have seen, his classes can be regarded as representatives of *predicates*, and some of these classes may themselves be represented by sets. The axioms of set theory would then take the form of conditions on how individuals may represent classes.⁸⁹

It is interesting to see how such representation may be combined with a formal theory of manifolds as already presented. As will become clear, there are various possible ways in which representatives might be assigned. Looked at in this light, the different axiomatic set theories could be looked on not as different speculations as to what there is, but as alternative conventions, choice among which would be a matter of expediency rather than metaphysical anguish.

We shall not treat representation in detail, but sample a few of the leading ideas which would need to be developed in order to further the concept of sets as representatives.

The first point to note is that, assuming that the domain of individuals contains some fixed number α of individuals, by Cantor's diagonal argument, we should never have sufficient individuals at our disposal to represent, all distinctly, all the manifolds of individuals there are except in the trivial case when $\alpha = 1$, when it is true that $\alpha = 2^\alpha - 1$. So either every manifold gets an individual, but sometimes distinct manifolds get the same individual, as representative, or else not all manifolds are represented. This applies most obviously to finite domains: in a domain of 2 individuals there are 3 distinct manifolds, for instance.

Let us then introduce a new primitive relation ' \triangleleft ', where $a \triangleleft b$ is to be understood as meaning that *a represents b*. Now if any manifolds could represent others, we should trivially be able to use each manifold

as its own representative. But more interesting is the case where only individuals are representatives.

How is representation to be arranged? One obvious suggestion is that no manifold should have more than one representative:

$$r1. \forall xw\forall u(x \triangleleft u \ \& \ w \triangleleft u \supset x = w)$$

while a second is that no individual should represent two distinct manifolds:

$$r2. \forall x\forall uv(x \triangleleft u \ \& \ x \triangleleft v \supset u \simeq v)$$

These are in no sense metaphysical truths: they are stipulations. It would not be *false* for either of these not to hold, any more than it is false that there are two Senators to every State of the Union, or that the Queen is Head of more than one State. But we cannot combine r2 with universal representation, r3:

$$r3. \forall u\exists x(x \triangleleft u)$$

(except in the case of the one-member domain). For consider the manifold r defined as follows: $r := \{x \mid \exists u(x \triangleleft u \ \& \ x \notin u)\}$. Then on any domain with more than one member, r must exist, for suppose every representative were included in the manifold it represents. Then, since every manifold is represented, by r3, all three submanifolds of $\{s, t\}$ must have representatives in $\{s, t\}$, which they can only do if one of the representatives represents more than one manifold, contrary to r2. So r has at least one member. Suppose $s \triangleleft r$. By the theorem of comprehension, $s \in r \equiv \exists u(s \triangleleft u \ \& \ s \notin u)$. If $s \notin r$ then s satisfies the condition because r exists, so $s \in r$. But then s must satisfy the condition of being a representative of some manifold it is not a member of. By r2, this must be r , so $s \notin r$, a contradiction. This is an exact analogue of the Cantor-Russell diagonal argument, and makes the point made above without recourse to the cardinality of the domain except that it must be greater than 1.

So some restrictions on representation are necessary. It is usual to have representatives only for the smaller, more tractable classes. This is the way of ZF and NBG set theory. Or we could restrict representation of classes which are the extensions of conditions of the form $\{x \mid A\}$ to

cases where the syntactic form of the condition is of a certain simple kind. This is the way of Quine.⁹⁰ We might have an individual which does not represent any manifold. If we have *only* one such, then it could be regarded as an analogue of the empty set. This can be expressed thus:

$$r4. \exists x (\sim \exists u(x \triangleleft u) \& \forall w (\sim \exists u(w \triangleleft u) \supset x = w))$$

In such circumstances, every individual other than this one, which we shall call \emptyset , is a representative. This provides an analogy with so-called pure set theory, where there are no individuals which are not sets. It would not be too inappropriate to regard $\emptyset \triangleleft \wedge$ as true in such circumstances. Pure set theory seems an extraordinary artifice in normal set theory, but its analogue in representative theory is no more than a recipe for not wasting individuals by having them not represent.

We may now see what an analogue of a set of sets is. It is simply a representative of a manifold of representatives. A theory of types among representatives would be a recipe for partitioning representatives and other individuals so that there would be manifolds u_0, u_1, u_2 , etc. with u_0 comprising individuals not representing anything (*Urelemente*), u_1 comprising representatives of manifolds included in u_0 , with perhaps an extra non-representative to serve as an empty representative, u_2 comprising representatives of submanifolds of u_1 , and so on. If we wished to continue indefinitely we should need to be assured of an infinite supply of individuals. Such a proposal is quite restrictive: it does not allow mixing of types, and every representative of a singleton is distinct from, and one type higher than, the individual it represents. On a countable domain with finitely many *Urelemente* every representative allowed by the theory could be forthcoming: n for *Urelemente*, the next 2^n for first-order representatives, the next 2^{2^n} for second-order representatives, and so on. But notice that not every manifold of individuals in the domain gets represented: there are not enough individuals to go round. Even u_1 will have gaps in it if the domain and the *Urelemente* are both of the same transfinite cardinality.

Systems of set theory designed to serve as foundations for mathematics all have axioms of infinity. It is important to notice that no such axiom is included in our calculus of manifolds. If we require that \triangleleft be irreflexive:

$$r5. \forall x(x \triangleleft a \supset x \neq a)$$

and add the further recursive requirement

$$r6. \exists u \forall x (x \in u \supset \exists w (w \triangleleft x \ \& \ w \in u))$$

then this can only be satisfied on an infinite domain. In particular, if representation is single-valued, satisfying r1, then it may be considered a partial function, and for any manifold u which is represented, we may denote its unique representative by $[u]$. Then r6 may be expressed as

$$r6a. \exists u \forall x (x \in u \supset [x] \in u)$$

If u is any given manifold, let the manifold generated from u by taking representatives of its members, representatives of these representatives and so on, be designated $Z(u)$. Then if there is a null representative \emptyset , the manifold $Z(\emptyset)$ is the manifold $\{\emptyset, [\emptyset], [[\emptyset]], \dots\}$, which is of course Zermelo's model for the natural numbers, or rather, an analogue of it.

If u is represented, let us represent this fact by the predicate R:

$$Ru := \exists x (x \triangleleft u)$$

One sensible stipulation regarding representation is that it be closed under the taking of submanifolds:⁹¹

$$r7. \forall u (Ru \supset \forall v (v \in u \supset Rv))$$

Another is that whenever a number of manifolds are represented, so is their union:

$$r8. R(\cup u (Ru \ \& \ A(u/a)))$$

where A is some condition on manifolds. In particular, selecting the condition $a \simeq a$, r8 yields the result that the union of represented manifolds is represented, $R(\cup u Ru)$. This is a sort of closure condition. We can get another sort in the following way. Let S be the predicate 'is a representative':

$$Sx := \exists u (x \triangleleft u)$$

then we could require that all manifolds of representatives be represented:

$$r9. \forall u(u \in \{x|Sx\} \supset Ru)$$

We may set up relations among representatives analogous to those holding among sets in ordinary set theory. For instance let η, κ be relations defined as follows:

$$s \eta t := \exists u(t \triangleleft u \ \& \ s \in u)$$

$$s \kappa t := \forall x(x \eta s \supset x \eta t)$$

Then η and κ are analogues of the membership and subset relations respectively. However, $s \kappa t \ \& \ t \kappa s$ only entail $s = t$ if $r1$ and $r2$ are satisfied. We can formulate as a stipulation an analogue of the power set axiom as follows:

$$r10. \forall x(Sx \supset R(\{w|w \kappa x\}))$$

while an analogue of the axiom of regularity is

$$r11. \forall x(Sx \supset \exists w(w \eta x \ \& \ \sim \exists x'(x' \eta w \ \& \ x' \eta x)))$$

Given an infinite domain, single-valued representation and a null representative \emptyset , with this axiom we know that providing representatives are forthcoming at every stage, a manifold N such that (i) $\emptyset \in N$ (ii) $\forall x(x \in N \supset [\{x, [x]\} \in N])$ and no other members besides those required by (i) and (ii), would furnish an analogue of von Neumann's version of the natural numbers. It would be the manifold $\{\emptyset, [\emptyset], [\{\emptyset, [\emptyset]\}], \dots\}$, and the relation $s \eta t$ among its members would be the natural ordering $<$.

Enough has perhaps by now been said to suggest that mixing manifolds with representatives offers a reasonable promise for keeping distinct Russell's and Cantor's two concepts of class, while not incurring the burdens of a Platonic ontology.⁹²

§7 Concluding Remark

"Sets", says Quine, "are classes... 'set' is simply a synonym of 'class' that happens to have more currency than 'class' in mathematical con-

texts".⁹³ Waiving the temptation to ask why Quine of all people should speak of synonyms, we might ask what underlies the claim. It is, I think, that there is identity, or at least continuity, between the mathematical concept of set and the familiar intuitive notion of a class. Modern set theory attempts to bite off as much of Cantor's Paradise as possible without biting off contradictions. It is worth asking whether in the process it has not forgotten what a class really is.⁹⁴

Notes

¹ Black, 1971, Stenius, 1974. It was from Black's paper that I obtained the view that plural terms and sets are counterparts, although, as I later discovered, Russell had arrived at the same idea much earlier. The extent of my agreement and disagreement with Black and Stenius (who are by no means in complete accord) will become clear through this paper. While I find that on the whole, their destructive comments are more successful than their constructive proposals, it still seems to me that they have been somewhat unfair to the earlier tradition of set theory, strands of which, as I show, come close to solving the difficulties. It is perhaps a reflection on the ahistorical way in which set theory is read and taught today that such strands should have been so completely overlooked.

² Cf. the remarks on this in the previous essay.

³ Abandoning the 'only if' part leads to Leśniewski's theory of 'collective classes', i. e. mereology. This kind of class is precisely Russell's class as one, for which see below. Leśniewski distinguishes collective from distributive classes. The latter do obey the extensionality principle. In Leśniewski however this is not a special set theory, but just the logic of names. It is interesting that Leśniewski was led to collective classes by consideration of Russell's paradox, and took a class as being most naturally conceived as the mereological sum. In view of the problems of the trinitarians, this is a natural attitude for anyone with nominalist inclinations. However, the calculus of manifolds, which I contend captures the notion of *class* rather than that of *whole*, bears affinities with Leśniewski's calculus of names, or 'ontology'. It also contains nothing a nominalist could find offensive.

⁴ Hence Leonard and Goodman's version of mereology, which they call the "calculus of individuals", might be thought well-titled. I am not convinced however, that masses of stuff (including limitlessly dispersed masses), which are amenable to mereological treatment, indeed cry out for it, are most aptly called 'individuals', since this term seems to apply most naturally to things falling under count concepts, whereas stuff falls under mass concepts. If there were a special grammatical form for mass nouns, distinguishing them from singular count nouns, then we should I think be far less inclined to heap masses and individuals together. However, this is a point with far-reaching consequences and ramifications, and cannot be pursued here. It should be emphasised that 'manifold' is to be understood in this paper as comprehending both individuals and pluralities. There is no difference between an individual and a single-membered manifold: the member *is* the manifold.

⁵ The predicate in manifold theory most closely analogous to '<' is not '∈' but '∈'. The manifold-theoretic notion of an individual is analogous to the mereological notion of

an atom. But manifold theory and mereology part company over this notion, for, if there are to be manifolds, there must be individuals (which might be called relative atoms) to comprise them, whereas the existence of composite entities does not, *pace* Leibniz, Wittgenstein etc., entail that there must be absolute atoms, i. e. entities without proper parts.

⁶ The relation '∈' is one example of a predicate which is, in the terminology of the previous essay, not perfectly distributive. More precisely, the predicate ' $a \in \xi$ ' does not distribute over manifolds, because from ' $a \in b$ ' and ' $c \in b$ ' it does not follow that ' $a \in c$ '. It is also clear that the relation '∈' is an ideal or *formal* relation, in the sense that 'exists' is a formal property, i. e. corresponds to no material property in the thing(s) concerned. In Kantian terms, '∈' is "no real predicate", arises simply from *as* being among the things designated by '*b*', for instance.

⁷ With Zermelo's axiomatisation, set theory became just another mathematical theory, albeit a very basic one. But the logicist intuition that in some sense 'class' is a fundamental *logical* notion, not a general mathematical one, deserves a better run for its money, provided, naturally, that the intuition can be separated from the familiar paradoxes.

⁸ Letter to Dedekind, Cantor, 1899. So when Mostowski, 1966, p. 141 speaks of Cantor distinguishing between consistent and inconsistent *sets*, this is seeing Cantor through the eyes of von Neumann and Gödel. In fairness to Mostowski, Cantor occasionally talks of consistent pluralities *being* (rather than forming) sets, but it is also clear from the context that this is loose talk.

⁹ The contrast with one such 'working mathematician', Felix Hausdorff, could not be greater. In his justly celebrated book, Hausdorff, 1914, he passes the paradoxes by with a cursory wave. In a recent article, Moore, 1978, G. H. Moore has shown convincingly how Zermelo's attitude was also that of a working mathematician, and that he was spurred to axiomatise set theory not to lay the ghost of the paradoxes but to provide a convincing proof of the well-ordering theorem using as weak a choice principle as possible, to gain the assent of the community of mathematicians, who had remained unconvinced by his earlier proof. For more on the weak principle, see § 4 below.

¹⁰ In his letter to Dedekind, Cantor suggested the following principles:

(1) Two equinumerous pluralities are either both inconsistent or both consistent (Cantor in fact says, 'are both "sets"', which is an example of the sort of remark mentioned at n. 8 above).

(2) Wherever we have a set of sets, the elements of these sets again form a set (*not* loose talk). (Union principle.)

(3) Every sub-plurality of a set is a set.

The first property was made in von Neumann, 1925–6, a characteristic of the difference between sets and ultimate classes: an ultimate class (to use Quine's felicitous term) is one which is equinumerous with the class of all sets, which cannot be a set, by Cantor's diagonal argument (as Cantor recognized).

¹¹ Russell, 1903, § 74, p. 76.

¹² *Ibid.*, § 104.

¹³ *Ibid.*, § 74. Russell here suggests that the class as one may be identified with the whole composed of the terms of the class, cf. § 139. This has the effect of allowing that more than one class as many may correspond to the same class as one. It also runs into difficulties about heaping together pluralities whose members come from widely different ontological domains.

¹⁴ Russell changed his mind, between writing about classes in the body of the book, probably in 1900–1, and writing the Appendix on Frege, late in 1902, about the strength of the Peano-Frege argument. My sympathies are, as I hope is clear, with his first thoughts.

¹⁵ *Ibid.*, § 104.

¹⁶ *Ibid.*, § 74.

- ¹⁷ Ibid., § 71. Note the widespread use of the concept of 'Und-Verbindung' by psychologists of the period, e. g. in Husserl, 1891a, p. 75f, or in the essay by Reinach below, § 15.
- ¹⁸ Ibid., §§ 70, 74, 490. Russell however does I think distinctly favour the idea of there being propositions with more than one subject. It may be that there is interference between the linguistic idea of a single subject-expression, and the semantic idea of a proposition's being *about* one or many things. Even a relational predication is *about* more than one thing, but unless the relation is expressed conjunctively (cf. the previous essay) these things are not all designated by one and the same subject-expression.
- ¹⁹ Ibid., § 70.
- ²⁰ Ibid., § 486.
- ²¹ Ibid., § 489.
- ²² Ibid., Appendix B. What is ironical about this is that the theory of types in the body of the book is motivated solely as a distinction between ones and many's, and rests on there being certain things which can be said of ones which cannot be said of many's and vice versa (§ 104). But it is of the essence of many's that they cannot be *members* of any class (ibid.), whereas all classes in the theory of types may be members of classes of the next higher type. So the theory of types enters at the expense of the one/many distinction, though it enters on the back of that distinction. There is therefore no justification for an *infinite* type hierarchy (§ 490), or even classes of classes.
- ²³ Cf. Miriamoff, 1917, Fraenkel, 1922.
- ²⁴ Von Neumann, 1925–6. The treatment is conducted entirely in terms of functions, but later commentators almost invariably present it more conventionally.
- ²⁵ In Bernays, 1937–54, these are symbolised '∈' for set membership, and 'η' for class membership. In § 6 we use the same pair of symbols in what is effectively the opposite way round.
- ²⁶ Bernays and Fraenkel, 1958, 41–2.
- ²⁷ Russell, 1903, § 489. The idea of representatives is further examined in § 6 below. Bernays also speaks of a set as *representing* a class in his 1937–54. A set *a* represents class *A* when $\forall x(x \in a \equiv x \eta A)$. It is a consequence of his axioms that every set represents a unique class, but of course not every class is represented by a set.
- ²⁸ This can be seen in part by the circumstance that Bernays does not quantify over classes, preferring always class parameters (free variables). Levy, 1973, p. 196 describes the move as one of replacing the metamathematical notion of a condition by the mathematical one of a class, while in the preface to his 1976, Müller reports that, unlike von Neumann, Bernays did not regard classes as real mathematical objects (p. vii). Levy describes this reluctance as 'not taking classes seriously', 1976, p. 205. That others have 'taken classes seriously', to the extent not only of quantifying over them and defining them impredicatively, but even considering their being elements of 'hyperclasses' – none of this can be laid at the feet of Bernays, who is always on stronger ground philosophically than those writers who block membership *solely* to prevent paradoxes from arising.
- ²⁹ Cf. my 1978. Other writers to "take common nouns seriously" include Lewis, 1970. A predicate is, after all, a sentence save some names (terms): if common nouns were predicates, then '*John man' should be a sentence, and if they were proper names, '*Tree is rotten' would be an acceptable sentence of English. The situation may not be so clear with other languages, but in English there is a clear syntactic difference between proper and common noun categories. Cf. the fuller remarks in the text below.
- ³⁰ It is interesting in this connection that Leśniewski's Ontology is often (and in my view preferably) called a calculus of names. Cf. § 5 below.
- ³¹ Cf. my 1978. Although predicate logic was developed primarily to answer the sentence-forming requirements of mathematicians, it is noticeable that mathematical texts no more avoid common nouns than other natural-language works. But since the official

formal syntax of modern mathematics does not use common nouns, their role is in part assumed by set-theoretic expressions. After so many years of familiarity with formal languages there is no reason why a fully adequate formalisation of noun-using mathematical language cannot be devised. No attempt has been made in this paper to do so, for this would involve greater complexity and unfamiliarity. Also, the arguments for accepting manifolds are independent of the use of common noun expressions.

³² Black, 1971, p. 104 in the reprint.

³³ 'Group' is to be taken here neither in the sense of McTaggart, 1921/7 nor that of Sprigge, 1970, nor, of course, in the mathematical sense.

³⁴ Stenius, 1974.

³⁵ Biologists evade the problem neatly by distinguishing between *Protozoa*, single-celled animals, *Metazoa*, multi-celled animals with two layers of cells, and sponges, which are set on one side as *Parazoa*.

³⁶ Such a view has to treat the identity of groups or individuals in flux as somehow second-rate. An obvious alternative, but one to be examined gingerly, is the view that there is also, or only, sortal-relative identity. Cf. Wiggins, 1967 or Griffin, 1977. In view of the distinctions between individuals, groups, wholes and classes made here I am hopeful that no such drastic expedient will be necessary.

³⁷ The view that the only genuine objects are those which can neither gain nor lose parts has a long history: it can be found in Leibniz and Hume, and has been defended most vigorously under the title of 'mereological essentialism' by Chisholm, e. g. 1976. For a rebuttal of this view, see Wiggins, 1979.

³⁸ It does of course entail a disjunction. Whether or not plural reference is eliminable, it is certainly useful. In any case, theoretical eliminability of certain kinds of expression, whether names, or variables, appears to me to carry ontological consequences only if it is supposed that ontology can be in some way "read off" linguistic facts.

³⁹ Cf. Russell, op. cit., §§ 68, 79, on 'is/are among'.

⁴⁰ Cf. § 5 below.

⁴¹ Russell, *ibid.*, § 74. But cf. his back-peddalling at § 491.

⁴² Cf. the remarks at n. 22 above. It is arguable that what Russell understood under the term 'theory of types' underwent changes, apart from the obvious one of the introduction of ramification, between 1903 and 1908. In that time, Russell was not always enamoured of the type-theoretic way out, advocating, not always at different times, at least three alternatives: the 'limitation of size' theory, anticipating ZF and NBG axiomatics, the 'zigzag' theory, anticipating Quine's NF, 1937, and most radically of all, the 'no class' theory, which took class expressions as incomplete symbols (Russell, 1973). Nothing illustrates more vividly the fecundity of Russell's intellect during this period than the apparent ease with which he could throw off radically new ideas.

⁴³ Cf. Russell, op. cit., § 71.

⁴⁴ As noted in n. 4, the concept of manifold includes also individuals; the word *plurality* is used for manifolds with more than one member. No single term bridges the gap between individuals and pluralities very well so the term 'manifold' is as good as any. But, so long as it is understood that the term is not understood as in the recent philosophical logical tradition, the term 'class' may be substituted for 'manifold' by anyone who finds the latter term barbaric.

⁴⁵ On the difference between 'anything' and 'everything' cf. my 1978.

⁴⁶ On syntactic connection cf. Husserl, LU IV, Ajdukiewicz, 1935, and other texts on categorial grammar, such as Lewis, 1970 or Cresswell, 1973.

⁴⁷ Just how messy can be seen by consulting Routley and Goddard, 1973.

⁴⁸ There are in fact two possible neutral identity predicates, one carrying, the other not carrying, existential import. Cf. the definition of ' \cong ' in § 4 below.

⁴⁹ Cf. Bernays and Fraenkel 1958, p. 49. But Bernays' solution is as artificial as Frege's.

- ⁵⁰ Frege, 1893, § 11.
- ⁵¹ That giving up bivalence may not be irremediably problematical may be seen by consulting e. g. Humberstone and Bell, 1977. But complications of the sort their proposals involve ought to be resisted unless they are forced upon us.
- ⁵² See e. g. the introduction of the null set in Hausdorff, 1914.
- ⁵³ See Routley, 1966.
- ⁵⁴ See e. g. Henry, 1972, Part II.
- ⁵⁵ On the interpretation of the quantifiers in Leśniewski see e. g. Küng, 1977. Orenstein, 1978 has disputed Küng's contention that Leśniewski's quantifiers are not substitutional (Appendix B), but it turns out that 'substitutional' has more than one possible meaning. At any rate, the quantifiers are certainly not objectual in Quine's sense.
- ⁵⁶ See Van Fraassen, 1966, Van Fraassen and Lambert, 1967.
- ⁵⁷ As e. g. in Thomason, 1970, Ch. V, § 5.
- ⁵⁸ See Quine, 1940, §§ 14, 16.
- ⁵⁹ See Church, 1956.
- ⁶⁰ For a convincing defence of this, cf. Hintikka, 1959.
- ⁶¹ Schröder uses the symbol ' \in ' in his 1890–1905, and the symbol remained in use for some time afterwards, e. g. with Löwenheim and Zermelo, but then dropped out in favour of ' \subset ' or, more usually today, ' \subseteq '. Schröder designed it as a combination of a sign for identity and one for proper inclusion. We do not use it in that sense, since for us ' $a \in b$ ' is only true when ' a ' is not empty. It can be seen more as a generalization of the sign ' \in ' for membership or singular inclusion to all cases of non-empty inclusion, proper or improper.
- ⁶² Quine 1963, § 4.
- ⁶³ It will be noticed that the axiom of choice, a16 below, is in fact an axiom *schema*, since it uses predicate parameters. In this, the theory resembles ZF.
- ⁶⁴ Cf. Stenius, 1974. It seems to me that Stenius is here rather bent on preserving as much of orthodox set theory from the flames as possible. Black, too, seems to be too ready to allow orthodox set theory as a legitimate development of the naive theory suggested by plural reference, rather than as embodying distortions leading away from the original intuitions.
- ⁶⁵ Stenius, in his endeavour to pick up Cantor's result that to any set there are 2^n subsets if the set has n members, overlooks the other possible subsets of the power set, although more general "second-order" counting procedures could be added to his to allow for these.
- ⁶⁶ It is still preferable to treat identity separately first. A similar-looking metatheorem is $\vdash a \approx \mathbf{1}(u \in a)$, which identifies every set with its power set.
- ⁶⁷ Zermelo, 1908.
- ⁶⁸ Zermelo, 1904.
- ⁶⁹ Zermelo, 1908.
- ⁷⁰ See Moore, 1978.
- ⁷¹ Hints that the disjoint choice principle, which Russell called the multiplicative axiom, might, in an environment of axioms for set theory weaker than, say, ZF, be strictly weaker than the full axiom of choice, arise out of various oddities in set theory. For example the proposition that every Boolean algebra has a maximal ideal, which is equivalent to Stone's representation theorem for Boolean algebras, has to date only been proved using the axiom of choice. But it is known (Halpern and Lévy, 1971) that the prime ideal theorem does not entail the principle of choice. It is notable that in our interpretation of what 'class' means, the general principle can only be stated using the concept of a function, while the weaker principle uses only the more general notion of a predicate or condition.
- ⁷² Schröder, 1890–1905. Cf. Church, 1976.

⁷³ Church, *ibid.*

⁷⁴ There is indeed a considerable difference between a thing's falling under a concept and a thing's being included in a class. Frege was quite right to insist that the latter must be separated from the subordination of one concept to another, but there is nothing wrong in treating membership as singular inclusion.

⁷⁵ As Schröder says (p. 245), 'hier wäre dann alles "wurst".'

⁷⁶ Russell, 1903, § 73.

⁷⁷ Cf. the final section of Frege's review of Schröder, 1890–1905, Frege, 1895.

⁷⁸ Interesting discussion comes from a perhaps unexpected quarter in G. E. Moore's *Commonplace Book* (Moore, 1962), pp. 13–4, where Moore discusses *class* and *extension*. He denies that with the ordinary meaning of 'class', classes could have less than two members, but that *if* we take ' $\forall x(\phi x \equiv \psi x)$ ' to imply ' ϕ and ψ have the same extension', then we must allow extensions having one or no members, so if we further identify classes with extensions, we must allow this for classes too. Moore seems very ready to throw over Russell's theory of classes on the strength of this somewhat grammatical point, and flirts with taking classes as pluralities, but in the end the discussion is inconclusive.

⁷⁹ In fact, because of the treatment of empty descriptions, it is the system FD_2 of Van Fraassen and Lambert, 1967.

⁸⁰ See Simons, 1978.

⁸¹ For an exposition of Leśniewski's Ontology, including the notions here interpreted, see either Lejewski, 1958 or Henry, 1972.

⁸² Asenjo, 1977b, takes Leśniewski *not* to have a set theory, but our disagreement with this is only a matter of how 'set' is to be interpreted.

⁸³ As e. g. is outlined briefly in my, 1978. Cf. n. 31 above.

⁸⁴ See e. g. Stoll, 1974, p. 214. (In the 2nd edition Stoll drops as redundant the requirement of distributivity.)

⁸⁵ I am indebted for some of the stimulus to writing this section to Wolfgang Degen, who is developing in detail a family of formal systems embodying alternative strategies for representation. Where I have concentrated my attention on the representation of classes only, Degen's work provides for the representation of predicate-entities in general. The Schröderian tradition and the idea of representation put forward in Löwenheim's 1940 were brought to my attention by Barry Smith; cf. his 1978.

⁸⁶ Stenius, *op. cit.*

⁸⁷ Frege, 1893, § 3.

⁸⁸ Löwenheim, 1940.

⁸⁹ Cf. Bernays' review of Löwenheim, Bernays, 1940.

⁹⁰ Quine, 1937.

⁹¹ Cf. the similar principle of Cantor, 1899, n. 10 above.

⁹² Goodman, 1977, indeed *defines* Platonism as the acceptance of classes in one's ontology.

⁹³ Quine, 1963, pp. 1, 3.

⁹⁴ I should like to thank David Bell and Barry Smith, and an anonymous referee of the *Journal of Philosophical Logic*, for comments on an earlier effort which helped me to make many improvements embodied in this essay. In at least one respect, I am conscious that more needs to be said, for nothing in this essay deals with the problem posed by vague predicates. Zermelo, 1908, was criticised for employing the unclear notion of a definite property. It must be said that most of what I have said in this essay was said without thought for what difference it might make if some properties entering into the formalism are not, in a suitable sense, definite. Are there vague groups and manifolds, or is this simply an unwarranted transference of an idea from the linguistic to the ontological sphere? I am heartened by the fact that we talk about vague groups, or at least

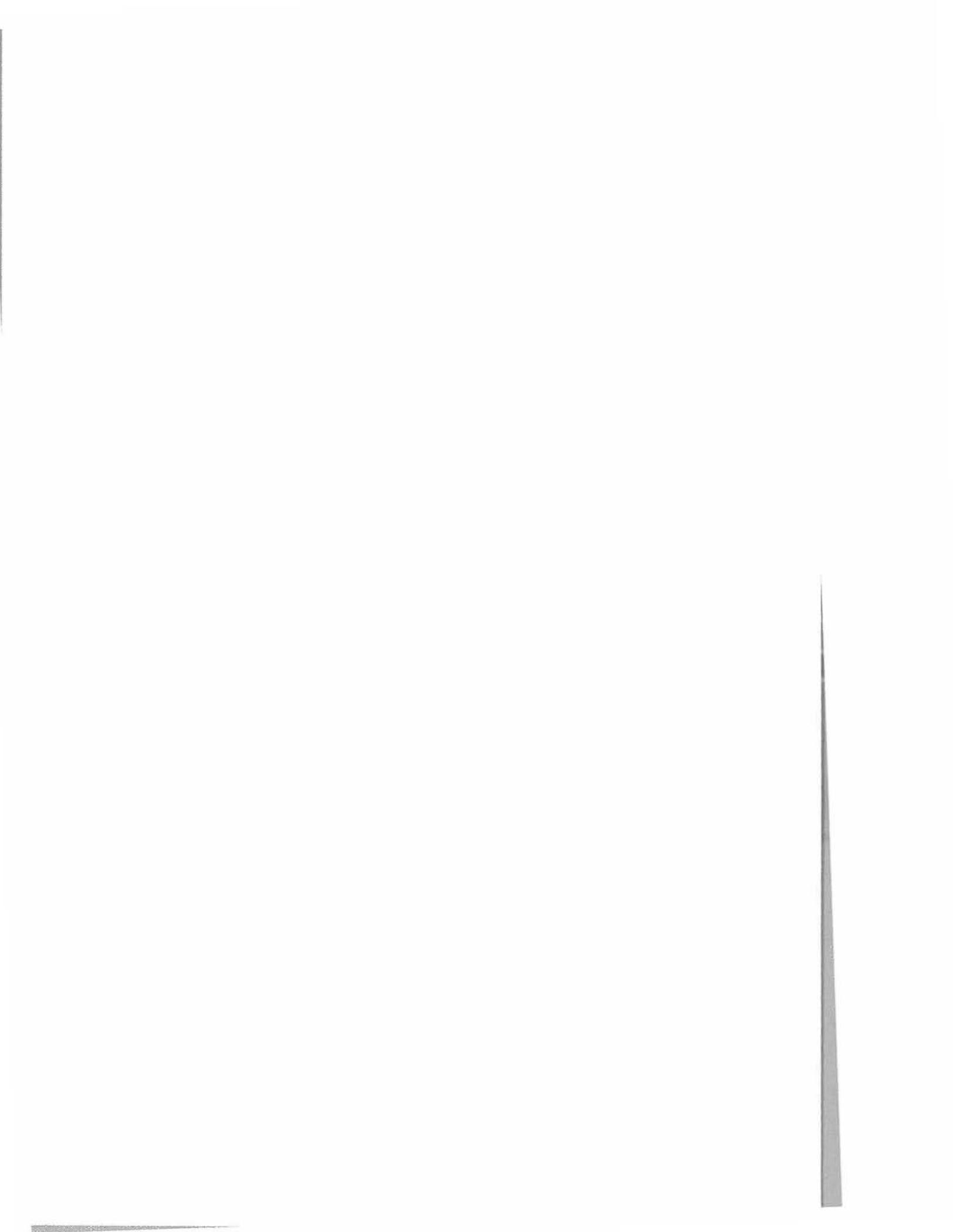
talk vaguely about groups, all the time in ordinary discourse, e. g. 'the trees in Austria', 'the utensils in this room'. This question will need separate consideration, but it cannot be offloaded as 'not our problem', as effectively happened with Zermelo set theory, as modified by Fraenkel, 1922 or Skolem, 1929.

I should also like to thank Prof. Karel Lambert for stimulating discussion of my ideas at a later stage. I owe to him correction of certain factual errors regarding free logic.

Bibliography

- Ajdukiewicz, K. 1935: "Die syntaktische Konnexität", *Studia Logica*, 1, 1–27, English translation, "Syntactic connection", in McCall, 1967, 1–27.
- Anderson, A. R. and Belnap, N. D. 1975: *Entailment: the logic of relevance and necessity*, Vol. I, Princeton: Princeton U. P.
- Anscombe, G. E. M. and Geach, P. T. 1961: *Three philosophers*, Oxford: Blackwell.
- Aquinas, St. T. 1964–76: *Summa Theologiae*, Blackfriars: London.
- Aristotle 1928: *Categories*, in *Works*, ed. W. D. Ross, Vol. I, Oxford: Oxford U. P.
1930: *De Caelo*, in *Works*, ed. W. D. Ross, Vol. II, Oxford: Oxford U. P.
- Armstrong, D. M. 1978: *Universals and scientific realism*, 2 vols., Cambridge: Cambridge U. P.
- Arruda, A. I. et al., eds. 1977: *Non-classical logics, model theory and computability*, Amsterdam: North-Holland.
- Asenjo, F. G. 1977a: "Formalizing multiple location", in Arruda, 1977, 25–36.
– 1977b: "Leśniewski's work and non-classical set theory", *Studia Logica*, 36, 249–55.
- Benacerraf, P. 1965: "What numbers could not be", *Philosophical Review*, 75, 47–73.
- Berg, J. 1962: *Bolzano's Logic*, Stockholm: Almqvist & Wiksell.
- Bernays, P. 1937–54: "A system of axiomatic set theory", *Journal of Symbolic Logic*, 2, 1937, 65–77; 6, 1941, 1–17; 7, 1942, 65–89; 133–45; 8, 1943, 89–106; 13, 1948, 65–79; 19, 1954, 81–96. Reprinted in Müller, ed., 1976, 1–120.
– 1940: Review of Löwenheim, 1940, *Journal of Symbolic Logic*, 6, 127–8.
- Bernays, P. and Fraenkel, A. A. 1958: *Axiomatic set theory*, Amsterdam: North-Holland.
- Black, M. 1971: "The elusiveness of sets", *Review of Metaphysics*, 24, 614–36, reprinted in Black, 1975, 85–108.
– 1975: *Caveats and critiques*, Ithaca: Cornell U. P.
- Bolzano, B. 1837: *Wissenschaftslehre: Versuch einer ausführlichen und größtenteils neuen Darstellung der Logik, mit steter Rücksicht auf den bisherigen Bearbeiter*, 4 vols., Sulzbach: Seidel. Partial English translation, *Theory of science*, ed. & trans., R. George, Oxford: Blackwell, 1972.
– 1851: *Paradoxien des Unendlichen*, Leipzig: Reclam. English translation, *Paradoxes of the infinite*, trans., D. A. Steele, London: Routledge, 1950.
- Breitkopf, A. 1978: "Axiomatisierung einiger Begriffe aus Nelson Goodmans *The Structure of Appearance*", *Erkenntnis*, 12, 229–47.
- Cantor, G. 1895–7: "Beiträge zur Begründung der transfiniten Mengenlehre", *Mathematische Annalen*, 46, 1895, 481–512; 49, 1897, 207–246. Reprinted in Cantor, 1932, 282–356. English translation, *Contributions to the founding of the theory of transfinite numbers*, by P. E. B. Jourdain, Chicago: Open Court, 1915.
– 1899: Cantor an Dedekind, in Cantor, 1932, 443–7, 451. English translation, "Letter to Dedekind" in Van Heijenoort, 1967, 113–7.

- 1932: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. E. Zermelo, Berlin: Springer, reprinted Hildesheim: Olms, 1966.
- Carnap, R. 1954: *Einführung in die symbolische Logik, mit besonderer Berücksichtigung ihrer Anwendungen*, Wien: Springer. English translation, *Introduction to symbolic logic and its applications*, by W. H. Meyer and J. Wilkinson, New York: Dover, 1958.
- Chisholm, R. M. 1976: *Person and object: a metaphysical study*, London: Allen & Unwin.
- 1978: "Brentano's conception of substance and accident" *Grazer Philosophische Studien*, 5, 197-210. This number also published as Chisholm & Haller, 1978.
- Chisholm, R. M. and Haller, R., eds. 1978: *Die Philosophie Franz Brentanos*, Amsterdam: Rodopi.
- Church, A. 1956: *Introduction to mathematical logic*, 2nd ed., Vol. I, Princeton: Princeton U. P.
- 1976: "Schröder's anticipation of the simple theory of types", *Erkenntnis*, 10, 407-11.
- Cresswell, M. C. 1973: *Logics and languages*, London: Methuen.
- Davidson, D. and Harman, G., eds. 1972: *Semantics of natural language*, Dordrecht: Reidel.
- Dummett, M. A. E. 1973: *Frege: philosophy of language*, London: Duckworth.
- Evans, G. 1977: "Pronouns, quantifiers, relative clauses I", *Canadian Journal of Philosophy*, 7, 467-536.
- Evans, G. and McDowell, J., eds. 1976: *Truth and meaning: essays in semantics*, Oxford: Clarendon.
- Findlay, J. N. 1963 *Meinong's theory of objects and values*, 2nd. ed., Oxford: Clarendon.
- Fogelin, R. J. 1976: *Wittgenstein*, London: Routledge.
- Fraenkel, A. A. 1922: "Über den Begriff 'definit' und die Unabhängigkeit des Auswahlaxioms", *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phil.-Math. Kl.*, 253-7.
- Fraenkel, A. A. et al. 1973: *Foundations of set theory*, 2nd ed., Amsterdam: North-Holland.
- Frege, G. 1884: *Die Grundlagen der Arithmetik, eine logisch-mathematische Untersuchung über den Begriff der Zahl*, Breslau: Köbner. English translation, *The foundations of arithmetic*, by J. L. Austin, Oxford: Blackwell, 2nd ed. 1953.
- 1893: *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, Bd. I, Jena: Pohle. Partial English translation, *The basic laws of arithmetic: exposition of the system*, by M. Furth, Berkeley: University of California Press., 1964.
- 1895: "Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik", *Archiv für systematische Philosophie*, 1, 433-56. Reprinted in Frege, 1976a, 92-112. English translation, "A critical elucidation of some points in E. Schröder's *Algebra der Logik*", in Frege, 1952, 86-106.
- 1952: *Translations from the philosophical writings of Gottlob Frege*, ed. and trans. P. T. Geach and M. Black, Oxford: Blackwell.
- 1976a: *Logische Untersuchungen*, G. Patzig ed., 2. Aufl., Göttingen: Vandenhoeck & Ruprecht. English translation of "Der Gedanke", "Die Verneinung" and "Gedankengefüge" by P. T. Geach & R. H. Stoothoff, *Logical Investigations*, Oxford: Blackwell, 1977.
- 1976b: *Wissenschaftliche Briefwechsel*, hrsg. G. Gabriel et al., Hamburg: Meiner. For an English translation of the Frege-Husserl correspondence, see Mohanty, ed. 1974.
- Geach, P. T. 1962: *Reference and generality: an examination of some medieval and modern theories*, Ithaca: Cornell U. P.
- Ginsberg, E. 1929: "Zur Husserlschen Lehre von den Ganzen und Teilen", *Archiv für systematische Philosophie*, 32, 109-20.
- Gödel, K. 1940: *The consistency of the axiom of choice and the generalized continuum-hypothesis with the axioms of set theory* (*Annals of Mathematical Studies*, No. 3), Princeton: Princeton U. P. Rev. ed. 1951.



- Küng, G. 1977: "The meaning of the quantifiers in the logic of Leśniewski", *Studia Logica*, 36, 309–22.
- Kuratowski, C. 1920: "Sur la notion de l'ordre dans la théorie des ensembles", *Fundamenta Mathematica*, 2, 161–71.
- Lambert, K. 1970: *Philosophical problems in logic*, Dordrecht: Reidel.
– 1974: "Impossible objects", *Inquiry*, 17, 221–44. See also Van Fraassen & Lambert.
- Laycock, H. 1972: "Some questions of ontology", *Philosophical Review*, 81, 3–42.
- Leibniz, G. W. 1903: *Opusculum et fragmenta inédita*, ed. L. Couturat, Paris: Alcan.
- Lejewski, C. 1958: "On Leśniewski's ontology", *Ratio*, 1, 150–76.
- Leonard, H. S. and Goodman, N. 1940: "The calculus of individuals and its uses", *Journal of Symbolic Logic*, 5, 45–55.
- Lévy, A. 1973 "The role of classes in set theory", in Fraenkel et al., 1973, 119–153, also in Müller, ed., 1976, 173–215. Page references to the latter.
- Lewis, D. K. 1970: "General semantics", *Synthese*, 22, 18–67, reprinted in Davidson & Harman, 1972, 169–218.
- Locke, J. 1975: *An essay concerning human understanding*, ed. P. H. Nidditch, Oxford: Clarendon.
- Löwenheim, L. 1940: "Einkleidung der Mathematik im Schröderschen Relativkalkül", *Journal of Symbolic Logic*, 5, 1–15.
- Luschei, E. C. 1962: *The logical systems of Leśniewski*, Amsterdam: North-Holland.
- McCall, S., ed. 1967: *Polish Logic, 1920–39*, Oxford: Clarendon.
- McTaggart, J. M. E. 1921–7: *The nature of existence*, 2 vols., Cambridge: Cambridge U. P.
- Meinong, A. 1899: "Über Gegenstände höherer Ordnung und deren Verhältnis zur inneren Wahrnehmung", *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, 21, 181–271. Reprinted in *Gesamtausgabe*, Bd. II, ed. R. Haller, Graz: Akademische Druck- und Verlagsanstalt, 1971, 377–480.
- Miriamoff, D. 1917: "Les antinomies de Russell et de Burali-Forti et le problème fondamentale de la théorie des ensembles", *L'Enseignement Mathématique*, 19, 37–52.
- Mohanty, J. N., ed. 1974: Frege-Husserl Correspondence, *Southwestern Journal of Philosophy*, 5, 83–95.
- Moore, G. E. 1962: *Commonplace Book*, ed. C. Lewy, London: Allen & Unwin.
- Moore, G. H. 1978: "The origins of Zermelo's axiomatization of set theory", *Journal of Philosophical Logic*, 7, 307–29.
- Mostowski, A. 1966: *Thirty years of foundational studies*, Oxford: Blackwell.
- Müller, G. H. ed. 1976: *Sets and classes: on the work of Paul Bernays*, Amsterdam: North-Holland.
- Neumann, J. von 1925–6: "Eine Axiomatisierung der Mengenlehre", *Journal für reine und angewandte Mathematik*, 154, 219–40. English translation "An axiomatization of set theory" in Van Heijenoort, 1967, 393–413.
- Orenstein, A. 1978: *Existence and the particular quantifier*, Philadelphia: Temple.
- Pantin, C. F. A. 1968: *The relations between the sciences*, Cambridge: Cambridge U. P.
- Plato 1953: *Parmenides*, in *The Dialogues*, ed. and trans. B. Jowett, 4th ed., Oxford: Oxford U. P.
- Quine, W. V. O. 1937: "New foundations for mathematical logic", *American Mathematical Monthly*, 44, 70–80. A revised and expanded version in Quine, 1961, 80–101.
– 1940: *Mathematical logic*, Cambridge, Mass.: Harvard U. P.
– 1960: *Word and object*, Cambridge, Mass.: Harvard U. P.
– 1961: *From a logical point of view*, 2nd ed., Cambridge, Mass.: Harvard U. P.
– 1963: *Set theory and its logic*, Cambridge, Mass.: Harvard U. P., § 34 reprinted in Klemke 1970, 372–87.
– 1966: "Russell's ontological development", *Journal of Philosophy*, 63, 657–67. Reprinted in Klemke, 1970, 3–14.

- Routley, R. 1966: "Some things do not exist", *Notre Dame Journal of Formal Logic*, 7, 251–77.
- Routley, R. & Goddard, L. 1973: *The logic of significance and context*, Edinburgh: Scottish Academic Press.
- Russell, B. 1903: *The principles of mathematics*, London: Allen & Unwin.
- 1959: *My philosophical development*, London: Allen & Unwin.
- 1973: "On the substitutional theory of classes and relations" (written 1906), in *Essays in Analysis*, ed. D. Lackey, London: Allen & Unwin, 165–89.
- Schröder, E. 1890–1905: *Vorlesungen über die Algebra der Logik*, 3 vols., Leipzig: Teubner.
- Scott, D. ed. 1971: *Axiomatic set theory. Proceedings of the American Mathematical Society's Symposia in Pure Mathematics*, 13, Providence: A. M. S.
- Simons, P. M. 1978: "Logic and common nouns", *Analysis*, 38, 161–7.
- Skolem, T. 1929: *Über einige Grundlagenfragen der Mathematik, Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo. I. Mat.-Naturvid. kl.*, no. 4.
- Smith, B. 1978: "Frege and Husserl: the ontology of reference", *Journal of the British Society for Phenomenology*, 9, 111–25.
- Spiegelberg, H., ed. 1971: "From Husserl to Heidegger: excerpts from a 1928 Freiburg diary by W. R. Boyce Gibson", *Journal of the British Society for Phenomenology*, 2, 58–83.
- Sprigge, T. L. S. 1970: *Facts, words and beliefs*, London: Routledge.
- Stenius, E. 1974: "Sets", *Synthese*, 27, 161–88.
- Stoll, R. R. 1974: *Sets, logic and axiomatic theories*, 2nd ed., San Francisco: Freeman.
- Stumpf, C. 1873: *Über den psychologischen Ursprung der Raumvorstellung*, Leipzig: Hirzel. Reprinted Zandvoort: Bonset, 1965.
- Thomason, R. M. 1970: *Symbolic logic: an introduction*, London: Macmillan.
- Van Fraassen, B. C. 1966: "The completeness of free logic", *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 12, 219–34.
- Van Fraassen, B. C. and Lambert, K. 1967: "On free description theory", *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 13, 225–40.
- Van Heijenoort, J. ed. 1967: *From Frege to Gödel: a source book in mathematical logic, 1879–1931*, Cambridge, Mass.: Harvard U. P.
- Wiggins, D. 1967: *Identity and spatio-temporal continuity*, Oxford: Blackwell.
- 1976: "The *De Re* 'Must': a note on the logical form of essentialist claims", in Evans and McDowell 1976, 285–312.
- 1979: "Asymmetrical essential dependence and the nature of continuants", *Grazer Philosophische Studien*, 7, 297–315.
- Willard, D. 1974: "Concerning Husserl's view of number", *Southwestern Journal of Philosophy*, 5, 97–109.
- Wittgenstein, L. 1961: *Tractatus Logico-Philosophicus*, translated from *Logisch-philosophische Abhandlung* by D. F. Pears and B. F. McGuinness, London: Routledge. Original German edition in *Annalen der Naturphilosophie*, 1921.
- Zemach, E. M. 1974: "In defence of relative identity", *Philosophical Studies*, 26, 207–18.
- Zermelo, E. 1904: "Beweis, dass jede Menge wohlgeordnet werden kann", *Mathematische Annalen*, 59, 514–6, English translation in Van Heijenoort, 1967, 139–41.
- 1908: "Neuer Beweis für die Möglichkeit einer Wohlordnung" and "Untersuchungen über die Grundlagen der Mengenlehre I", *Mathematische Annalen*, 65, 107–28, resp. 261–81, English translation in Van Heijenoort 1967, 183–215.